

# TOROIDAL AND KLEIN BOTTLE BOUNDARY SLOPES

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**ABSTRACT.** Let  $M$  be a compact, connected, orientable, irreducible 3-manifold and  $T_0$  an incompressible torus boundary component of  $M$  such that the pair  $(M, T_0)$  is not cabled. By a result of C. Gordon, if  $(S, \partial S), (T, \partial T) \subset (M, T_0)$  are incompressible punctured tori with boundary slopes at distance  $\Delta = \Delta(\partial S, \partial T)$ , then  $\Delta \leq 8$ , and the cases where  $\Delta = 6, 7, 8$  are very few and classified. We give a simplified proof of this result (or rather, of its *reduction process*), using an improved estimate for the maximum possible number of mutually parallel negative edges in the graphs of intersection of  $S$  and  $T$ . We also extend Gordon's result by allowing either  $S$  or  $T$  to be an essential Klein bottle.

## 1. INTRODUCTION

Let  $M$  be a compact, connected, orientable, irreducible 3-manifold, and  $T_0$  an incompressible torus boundary component of  $M$ . If  $r_1, r_2$  are two slopes in  $T_0$ , we denote their *distance*, ie their minimum geometric intersection number in  $T_0$ , by  $\Delta(r_1, r_2)$ . By a *surface* we mean a compact 2-dimensional manifold, not necessarily orientable. A properly embedded surface in  $M$  with nonempty boundary which is not a disk is said to be *essential* if it is geometrically incompressible and boundary incompressible in  $M$ . We will use the notion of a *cabled pair*  $(M, T_0)$  in the sense of [3].

Let  $(F, \partial F) \subset (M, T_0)$  be a punctured torus. We say that  $F$  is *generated by a (an essential) Klein bottle* if there is a (an essential, resp.) punctured Klein bottle  $(P, \partial P) \subset (M, T_0)$  such that  $F$  is isotopic in  $M$  to the frontier of a regular neighborhood of  $P$  in  $M$ . We also say that  $F$  is  *$\mathcal{K}$ -incompressible* if  $F$  is either incompressible or generated by an essential Klein bottle. In this paper we give a proof of the following result.

**Theorem 1.1.** *Let  $(F_1, \partial F_1), (F_2, \partial F_2) \subset (M, T_0)$  be  $\mathcal{K}$ -incompressible tori, and let  $\Delta = \Delta(\partial F_1, \partial F_2)$ . If the pair  $(M, T_0)$  is not cabled then  $\Delta \leq 8$ , and if  $\Delta \geq 6$  then  $|\partial F_1|, |\partial F_2| \leq 2$ .*

The corollary below follows immediately from Theorem 1.1; along with [9, Theorem 1.2 and §6], it can be used to obtain the classification of the manifolds  $M$  that contain essential punctured Klein bottles with boundary slopes at distance  $\Delta \geq 6$ .

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**Corollary 1.2.** *Let  $(F_1, \partial F_1), (F_2, \partial F_2) \subset (M, T_0)$  be punctured essential Klein bottles, and let  $\Delta = \Delta(\partial F_1, \partial F_2)$ . If the pair  $(M, T_0)$  is not cabled then  $\Delta \leq 8$ , and if  $\Delta \geq 6$  then  $|\partial F_1| = 1 = |\partial F_2|$ , with  $\Delta = 6, 8$ .  $\square$*

Theorem 1.1 is well known when the surfaces  $F_\alpha$  are both tori, in which case it follows from the proof of [2, Proposition 1.5]. The case where both surfaces are Klein bottles has been discussed more recently in [6, Corollary 1.5] (for  $\Delta \geq 5$ ) and [8, Theorem 1.4] (for  $\Delta \geq 5$ ), under the added hypothesis that  $M$  is hyperbolic. Thus, for  $\Delta \geq 6$ , modulo the classification of the manifolds  $M$ , Theorem 1.1 and its corollary extend the range of applicability of [2, Proposition 1.5] to include the case of essential Klein bottles, and of [6, Corollary 1.5] and [8, Theorem 1.4] to allow for manifolds that may not be hyperbolic.

A general approach to the proof of results similar to Theorem 1.1 involves what we may call a *reduction process*, where, say, a condition on the distance between the boundary slopes, like  $\Delta \geq 6$ , creates ‘large’ families of parallel edges, whose presence may restrict the number of boundary components of at least one surface to be ‘small’, or the topology of  $M$  to be ‘degenerate’, in some sense. If the ‘small’ cases are sufficiently small, they can be dealt with separately or classified completely. In fact, for  $\Delta \geq 6$ , combining the classification of the pairs  $(M, T_0)$  in [2, Proposition 1.5] with Theorem 1.1 and [9, Theorem 1.2], it follows that there are exactly four manifolds  $(M, T_0)$  in Theorem 1.1, all obtained via Dehn fillings along one of the boundary components of the Whitehead link exterior, and that if  $\Delta = 6, 8$  and  $F_\alpha$  is a torus then  $F_\alpha$  is incompressible and generated by a once punctured Klein bottle.

In the proof of Theorem 1.1 we present here we use some fundamental results from the paper [3], with the addition of Lemma 2.1 [2, §2] (on parallelism of edges), the notion of jumping number [2, §2], and the parity rules from [1, 5, 7]; the new ingredients are contained in Proposition 3.4, the main technical result of this paper, which roughly states that if  $(M, T_0)$  is not cabled and contains two  $\mathcal{K}$ -incompressible tori  $(T, \partial T), (T', \partial T') \subset (M, T_0)$  with  $\Delta(\partial T, \partial T') \geq 1$ , then, for any surface  $S \subset M$  that intersects  $T$  in essential graphs, any collection of mutually parallel negative edges of the graph  $S \cap T \subset S$  has at most  $|\partial T| + 1$  edges, unless  $M$  is one of three exceptional toroidal manifolds, in which case  $\Delta(\partial T, \partial T') = 1, 2$  or  $4$ . We remark that the current best bound used in similar contexts is  $2 \cdot |\partial T|$ , for  $t \geq 4$  (cf [2, Corollary 5.5]). It is the use of the upper bound  $|\partial T| + 1$  of Proposition 3.4 that gives rise to a rather short reduction process for Theorem 1.1.

The paper is organized as follows. In Section 2 we present several basic definitions and facts related to the graphs of intersection produced by two surfaces in  $M$  with transverse intersection. Section 3 is devoted to the discussion of bounds for the sizes of collections of mutually parallel edges in the graphs of intersection of two surfaces in  $M$ ; the first two subsections deal with the case of positive edges and some known facts for the case of negative

edges, and the remaining two sections contain the proof of Proposition 3.4. Finally, the proof of Theorem 1.1 is given in Section 4.

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## 2. PRELIMINARIES

Let  $M$  be a compact, connected, orientable, irreducible 3-manifold with an incompressible torus boundary component  $T_0$ . For any nontrivial slope  $r \subset T_0$ ,  $M(r)$  will denote the Dehn filled manifold  $M \cup_{T_0} V$ , where  $V$  is a solid torus such that  $r$  bounds a disk in  $V$ . If  $F \subset M$  is a properly embedded surface and  $r$  is the slope of the circles  $F \cap T_0$ , then  $\widehat{F}$  will denote the surface in  $M(r)$  obtained from  $F$  by capping off any components of  $\partial F$  in  $T_0$  with disjoint meridian disks in  $V$ .

Let  $F_1, F_2$  be any two properly embedded surfaces in  $M$  (orientable or not) which intersect transversely in a minimum number of components; in particular, if  $r_\alpha$  is the slope of the circles  $\partial F_\alpha \cap T_0$  in  $T_0$ , and  $\Delta = \Delta(r_1, r_2)$ , then any two components of  $\partial F_1 \cap T_0$  and  $\partial F_2 \cap T_0$  intersect transversely in  $\Delta$  points.

We say that  $G_{F_1} = F_1 \cap F_2 \subset F_1$  and  $G_{F_2} = F_1 \cap F_2 \subset F_2$  are the *graphs of intersection* between  $F_1$  and  $F_2$ . Either of these graphs is *essential* if each component of  $F_1 \cap F_2$  is geometrically essential in the corresponding surface. The graph  $G_{F_\alpha}$  has *fat vertices* the components of  $\partial F_\alpha$  and edges the arc components of  $F_1 \cap F_2$ ; there may also be some circle components present. An edge of  $F_1 \cap F_2$  with both endpoints in  $T_0$  is called an *internal edge*.

Let  $n_1 = |\partial F_1 \cap T_0|$  and  $n_2 = |\partial F_2 \cap T_0|$ . We label the components of  $\partial F_\alpha \cap T_0$  as  $\partial_1 F_\alpha, \partial_2 F_\alpha, \dots, \partial_{n_\alpha} F_\alpha$ , consecutively in their order of appearance along  $T_0$  (in some direction), and then label each intersection point between  $\partial_i F_1$  and  $\partial_j F_2$  with  $j$  in  $G_{F_1}$  and  $i$  in  $G_{F_2}$ . In this way, any endpoint of an edge of  $F_1 \cap F_2$  that lies in  $T_0$  gets a label in each graph of intersection, and internal edges get labels at both endpoints.

Following [5, 7], we orient the components of  $\partial F_\alpha \cap T_0$  coherently on  $T_0$ , and say that an internal edge  $e$  of  $F_1 \cap F_2$  has a *positive* or *negative sign* in  $G_{F_\alpha}$  depending on whether the orientations of the components of  $\partial F_\alpha$  (possibly the same) around a small rectangular regular neighborhood of  $e$  in  $F_\alpha$  appear as in Fig. 1.

Alternatively (cf [1]), if  $F_\alpha$  is orientable, we fix an orientation on  $F_\alpha$ , induce an orientation on the components of  $\partial F_\alpha \cap T_0$ , and then say that two components of  $\partial F_\alpha \cap T_0$  have *the same parity* if their given orientations agree on  $T_0$ , and *opposite parity* otherwise. This divides the components of  $\partial F_\alpha \cap T_0$  into two parity classes, and we may call the vertices in one class *positive*, and the vertices in the other class *negative*. It is then not hard to see that an internal edge of  $F_1 \cap F_2$  is positive (negative) in  $F_\alpha$  iff it connects two vertices of  $G_{F_\alpha}$  of the same (opposite, resp.) parity. In this context, if the vertices  $\partial_i F_\alpha$  of  $F_\alpha$  are all of the same parity we will say that  $F_\alpha$  is

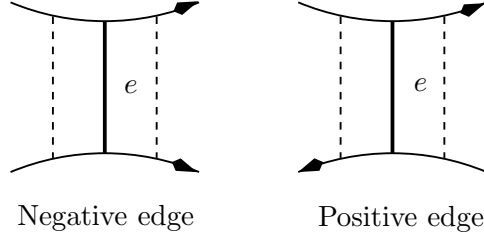


FIGURE 1.

*polarized*, and that it is *neutral* if there are the same number of vertices of either parity.

A collection of edges in  $G_{F_\alpha}$  whose union is a circle in  $\widehat{F}_\alpha$  (where the circle is constructed in the obvious way, by collapsing the vertices of  $F_\alpha$  on  $T_0$  into points in  $\widehat{F}_\alpha$ ) is called a *cycle*. A cycle in  $F_\alpha$  is *nontrivial* if it is not contained in a disk in  $\widehat{F}_\alpha$ . We call a cycle in  $F_\alpha$  consisting of a single edge a *loop edge*; notice that if  $F_\alpha$  is orientable then a loop edge in  $F_\alpha$  is positive.

Two edges of  $F_1 \cap F_2$  are said to be *parallel* in  $F_\alpha$  if they cobound a rectangular disk subregion in  $F_\alpha$ . Suppose that two internal edges  $e, e'$  of  $F_1 \cap F_2$  are positive, parallel, and consecutive in  $F_\alpha$ , and let  $F$  be the disk face in  $G_{F_\alpha}$  they cobound. We say that  $F$  is an *S-cycle face of type  $\{j, j+1\}$*  of  $G_{F_\alpha}$  (with  $j, j+1$  well defined mod  $n_\beta$ ) if the labels at the endpoints of each edge  $e, e'$  are  $j$  and  $j+1$ ; this is a restricted version of the more general notion of a Scharlemann cycle, which we will not use in this paper.

The following lemma summarizes several fundamental results we will use in the sequel.

**Lemma 2.1.** *Let  $F_1, F_2$  be properly embedded surfaces in  $M$  with essential graphs of intersection  $G_{F_1}, G_{F_2}$ .*

- (a) *Parity Rule ([1, 5, 7]): for  $\{\alpha, \beta\} = \{1, 2\}$ , an internal edge of  $F_1 \cap F_2$  is positive in  $G_{F_\alpha}$  iff it is negative in  $G_{F_\beta}$ .*
- (b) *Suppose  $(M, T_0)$  is not cabled and  $F_1, F_2$  are orientable. Then no two internal edges of  $F_1 \cap F_2$  are parallel in both  $G_{F_1}$  and  $G_{F_2}$  ([2, Lemma 2.5]), and if  $n_\alpha \geq 2$  and  $E$  is a family of mutually parallel, consecutive, internal negative edges in  $G_{F_\beta}$  then no component of  $F_\alpha \setminus \cup E$  is a disk in  $F_\alpha$  ([3]).*
- (c) *If  $(M, T_0)$  is not cabled,  $F_1$  is planar,  $F_2$  is toroidal, and  $\partial F_1, \partial F_2 \subset T_0$ , then  $\Delta \leq 5$  ([3]).*  $\square$

**2.1. Reduced graphs.** Let  $G$  be an essential connected graph on a compact punctured surface  $\mathcal{F}$ , of the type constructed above. We let  $V(G), E(G)$  denote the sets of (fat) vertices and edges of  $G$ , respectively. Cutting each edge of  $G$  along some interior point splits the edges into pieces which we call the *local edges* of  $G$ . The degree of a vertex  $v$  of  $G$ , denoted by  $\deg_G(v)$  or  $\deg(v)$ , is then the number of local edges of  $G$  that are incident to  $v$ .

For an integer  $k \geq 0$ , the notation  $\deg \geq k$  ( $\deg \equiv k$ ) in  $G$  will mean that  $\deg(v) \geq k$  ( $\deg(v) = k$ , resp.) holds for any  $v \in V(G)$ . Thus, the degree of any vertex  $\partial_i F_\alpha$  of  $G_{F_\alpha}$  is  $\Delta \cdot n_\beta$  and the labels  $1, 2, \dots, n_\beta$  repeat  $\Delta$  times in blocks consecutively around  $\partial_i F_\alpha$ .

Let  $N(E(G))$  be a small product neighborhood of  $E(G)$  in  $\mathcal{F}$ . Then the closure of any component of  $\mathcal{F} \setminus N(E(G))$  is called a *face* of  $G$ . Observe that if  $F$  is any face of  $G$ , then  $\partial F$  is a union of segments of the form  $e \times 0, e \times 1$ 's, called the *edges* of  $F$ , and segments coming from the  $\partial_i \mathcal{F}$ 's, called the *corners* of  $F$ . We call a disk face of  $G$  with  $n$  sides (and  $n$  corners) a *disk  $n$ -face*; disk 2-faces or 3-faces are also referred to as *bigons* or *triangles*, respectively.

The graph  $G$  is said to be *reduced* if no two of its edges are parallel. The *reduced graph*  $\overline{G}$  of  $G$  is the graph obtained by amalgamating any maximal collection of mutually parallel edges of  $G$  into a single edge. Notice that any disk face in a reduced graph is at least a triangle.

The next result gives two useful facts about reduced graphs on a torus.

**Lemma 2.2.** *Let  $G$  be a reduced graph on a torus with  $V$  vertices,  $E$  edges, and  $\deg \geq 1$ .*

- (a) *If  $\deg \geq 6$  in  $G$  then  $\deg \equiv 6$  in  $G$  and all faces of  $G$  are triangles.*
- (b) *If  $G$  has no triangle faces then  $G$  has a vertex of degree at most 4.*

*Proof.* Part (a) is well known (cf [2, Lemma 3.2]). For part (b), let  $d$  be the number of disk faces of  $G$  and set  $n = \min\{\deg(u) \mid u \text{ is a vertex of } G\} \geq 1$ . Then  $nV \leq 2E$ , and since any disk face of  $G$  is at least a 4-face then  $4d \leq 2E$ . Combining these relations with Euler's relation  $E \leq V + d$  then implies that  $n \leq 4$ , hence  $G$  has a vertex of degree at most 4.  $\square$

**2.2. Edge orbits and permutations.** We will denote any edge in the reduced graph  $\overline{G}_{F_\alpha}$  generically by the symbol  $\overline{e}$ . Hence,  $\overline{e}$  represents a collection  $e_1, e_2, \dots, e_k$  of mutually parallel, consecutive, same sign edges in  $G_{F_\alpha}$ , in which case we say that  $|\overline{e}| = k$  is the *size* of  $\overline{e}$ , and that the sign of  $\overline{e}$  is positive (negative) if all the edges in  $\overline{e}$  are positive (negative, resp).

Suppose that  $n_\beta \geq 2$ , and that  $E$  is a collection of  $n_\beta$  mutually parallel, consecutive internal edges of  $G_{F_\alpha}$ . We assume that these edges have endpoints in the vertices  $u_i, u_{i'}$  of  $G_{F_\alpha}$  (with  $u_i = u_{i'}$  allowed), and that all edges in  $\overline{e}$  are oriented to run from  $u_i$  to  $u_{i'}$  (the orientation is arbitrary if  $u_i = u_{i'}$ ). Then each of the labels  $1, 2, \dots, n_\beta$  appears exactly once at the endpoints of the edges of  $E$  at each of the vertices  $u_i$  and  $u_{i'}$ , and so the set  $E$  induces a permutation  $\sigma$  on the set  $\{1, 2, \dots, n_\beta\}$ , defined by matching the labels at the endpoints of the edges of  $E$  in  $u_i$  with the corresponding labels at the endpoints of these edges in  $u_{i'}$ . This permutation is of the form  $\sigma(x) \equiv \alpha - \varepsilon \cdot x \pmod{n_\beta}$ , where  $\varepsilon = +1, -1$  is the sign of the edges in  $E$  (see Figs. 2(a) and 3); reversing the orientation of the edges replaces  $\sigma$  with its inverse. Observe that if the edges in  $E$  are positive then  $\sigma^2 = \text{id}$ , and that  $\sigma \neq \text{id}$  whenever  $F_\beta$  is orientable by the parity rule.

More generally, it is not hard to see that if  $E'$  is any collection of mutually parallel, consecutive internal edges of  $G_{F_\alpha}$ , with  $|E'| \geq n_\beta$ , then any two subfamilies of  $E'$  with  $n_\beta$  consecutive edges induce the same permutation; we refer to this common permutation as the permutation *induced* by  $E'$ .

The union in  $G_{F_\beta}$  of all edges in  $E$ , along with all vertices of  $G_{F_\beta}$  at their endpoints, form a subgraph  $\Gamma_E$  of  $G_{F_\beta}$ ; we call any component of  $\Gamma_E$  an *edge orbit* of  $E$ . Each orbit of  $\sigma$  then corresponds uniquely to some edge orbit of  $E$ : for the labels of the vertices of  $G_{F_\beta}$  at the endpoints of the edges in an edge orbit of  $E$  form an orbit of  $\sigma$ .

**2.3. Strings.** We denote by  $I_{i,i+1}$  the annulus cobounded in  $T_0$  by the circles  $\partial_i F_\alpha, \partial_{i+1} F_\alpha$ , with labels  $i, i+1$  well defined mod  $n_\beta$ , and call it a *string* of  $F_\alpha$ .

Notice that the corners of any face of  $G_{F_\beta}$  are spanning arcs along some of the strings of  $F_\alpha$ . For  $F_\alpha$  an orientable surface, let  $N(F_\alpha) = F_\alpha \times [0, 1]$  be a small product regular neighborhood of  $F_\alpha$  in  $M$ ; if  $F$  is a face of  $G_{F_\beta}$ , we will say that  $F$  *locally lies on one side of  $F_\alpha$*  if  $F$  intersects only one of the two surfaces  $F_\alpha \times 0$  or  $F_\alpha \times 1$ .

**2.4.  $\mathcal{K}$ -incompressible tori.** Suppose that the punctured torus  $(T, \partial T) \subset (M, T_0)$  is generated by an essential punctured Klein bottle  $P \subset M$ , and that  $S \subset M$  is a properly embedded surface which intersects  $P$  in essential graphs  $G_{S,P} = S \cap P \subset S$  and  $G_P = S \cap P \subset P$ . Let  $N(P)$  be a regular neighborhood of  $P$  in  $M$ , and isotope  $T$  so that  $T = \text{fr } N(P)$ . For  $N(P)$  small enough, the intersection  $S \cap T$  will be transverse and the graphs  $G_{S,T} = S \cap T \subset S$  and  $G_T = S \cap T \subset T$  will also be essential; in fact, the graph  $G_{S,T}$  will be the frontier in  $S$  of the regular neighborhood  $N(P) \cap S$  of all the components of  $G_{S,P}$ . Moreover, if  $\bar{e}$  is an edge of  $\overline{G}_P$ , then  $\bar{e}$  gives rise to two distinct edges  $\bar{e}_1, \bar{e}_2$  in  $\overline{G}_T$ , each of the same size as  $\bar{e}$ , which are parallel in  $N(P)$ , and if  $\partial S \subset T_0$  and  $|\bar{e}| \geq |\partial S|$ , then the edges  $\bar{e}, \bar{e}_1$ , and  $\bar{e}_2$  all have the same sign and induce the same permutation.

In particular, if  $T_1, T_2$  are  $\mathcal{K}$ -incompressible tori in  $(M, T_0)$ , then it is possible to isotope  $T_1$  or  $T_2$  so that both graphs of intersection  $G_{T_1}$  and  $G_{T_2}$  are essential.

**2.5. S-cycles and Klein bottles.** In this section we assume that  $(T, \partial T) \subset (M, T_0)$  is a twice punctured torus and  $S$  is a properly embedded surface in  $M$  which intersects  $T$  in essential graphs  $G_S, G_T$ . In particular, all edges of  $S \cap T$  are internal, and if  $G_S$  has an S-cycle face then  $T$  is neutral by the parity rule.

The next result follows in part from the proof of [4, Lemma 5.2]; we include a sketch of its proof for the convenience of the reader.

**Lemma 2.3.** *Suppose that  $G_S$  has two S-cycle faces  $F_1, F_2$  which lie locally on the side of  $T$  corresponding to the string  $I_{1,2}$ , such that the circles*

$\partial F_1, \partial F_2$  are not isotopic in the closed surface  $T \cup I_{1,2}$ . Then  $T$  is generated by a once punctured Klein bottle  $P$  with  $\partial P \subset I_{1,2}$ , which is essential whenever  $(M, T_0)$  is not cabled and  $M(\partial T)$  is irreducible.

*Proof.* As observed above, the presence of S-cycle faces in  $G_S$  implies that  $T$  is neutral, hence the surface  $T \cup I_{1,2}$  is closed, orientable, and of genus two. Since the circles  $\partial F_1, \partial F_2$  intersect the string  $I_{1,2}$  each in one spanning arc, and are disjoint and not isotopic in  $T \cup I_{1,2}$ , compressing the surface  $T \cup I_{1,2}$  in  $M$  along the disks  $F_1, F_2$  produces a 2-sphere embedded in  $M$ , which bounds a 3-ball in  $M$  since  $M$  is irreducible. It follows that  $T$  separates  $M$  into two components with closures  $T^+, T^-$ , so that if  $T^+$  is the component containing the string  $I_{1,2}$  then  $T^+$  is a genus two handlebody with complete disk system  $F_1, F_2$ . Moreover, if  $x, y$  are generators of  $\pi_1(T^+)$  which are dual to  $F_1, F_2$ , respectively, then, with some orientation convention, if  $c$  is the core of  $I_{1,2}$  then  $c$  represents the word  $x^2 y^2$  in  $\pi_1(T^+)$ . As  $c$  intersects each disk  $F_1, F_2$  coherently in two points, it is not hard to see that  $c$  bounds a once punctured Klein bottle  $P$  in  $T^+$  such that  $T^+$  is homeomorphic to  $N(P)$ .

Finally, if  $M(\partial T)$  is irreducible then  $\hat{P}$  is incompressible in  $M(\partial T)$ , so  $P$  is incompressible in  $M$  since  $T_0$  is incompressible; and if  $P$  boundary compresses in  $M$  then it boundary compresses into a Moebius band, whence  $(M, T_0)$  is  $(1, 2)$ -cabled. The lemma follows.  $\square$

### 3. EDGE SIZE

In this section we will assume that  $(T, \partial T) \subset (M, T_0)$  is a punctured torus with  $t = |\partial T| \geq 1$  and  $S$  a properly embedded surface in  $M$  which intersects  $T$  in essential graphs  $G_S, G_T$ , and establish bounds for the sizes of the edges in the reduced graph  $\overline{G}_S$ , under suitable conditions. We denote the vertices  $S \cap T_0$  of  $G_S$  by  $u_i$ 's, and the vertices of  $G_T$  by  $v_j$ 's; notice that all edges in  $G_S$  are internal.

**3.1. Positive edges.** A bound for the size of a positive edge of  $\overline{G}_S$  can be easily found.

**Lemma 3.1.** *Suppose  $(M, T_0)$  is not cabled. If  $t \geq 3$  and  $\bar{e}$  is a positive edge of  $\overline{G}_S$  then  $|\bar{e}| \leq t$ , and if  $|\bar{e}| = t$  then  $t$  is even, the edge orbit of  $\bar{e}$  is a subgraph of  $\overline{G}_T$  isomorphic to the graph of Fig. 2 (thick edges only), and some vertex of  $\overline{G}_T$  has at most two incident positive nonloop edges.*

*Proof.* Let  $t \geq 3$  and  $\bar{e}$  be a positive edge of  $\overline{G}_S$  of size  $\geq t$ , with consecutive edges  $e_1, e_2, \dots, e_t, e_{t+1}, \dots$  labeled and running from  $u_i$  to  $u_{i'}$ , as shown in Fig. 2(a). The collection  $E = \{e_1, e_2, \dots, e_t\}$  then induces a permutation  $\sigma$  of the form  $x \mapsto \alpha - x$ , a nontrivial involution, so the edge orbits of  $E$  are a family of disjoint cycles of length 2, which are nontrivial in  $\hat{T}$  by Lemma 2.1(b), and hence the subgraph of  $G_T$  generated by these cycle edge orbits is isomorphic to the graph shown in Fig. 2(b) (thick edges only). In

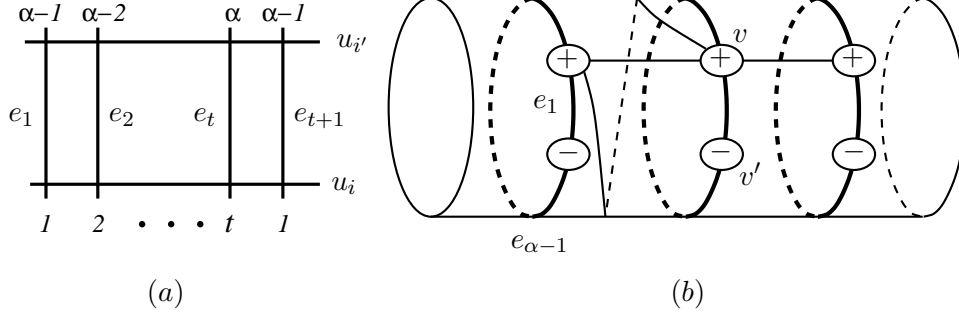


FIGURE 2.

particular,  $t$  is even, so  $t \geq 4$ , and there are  $t/2$  such cycles. Consider now the edges  $e_1, e_{\alpha-1}$ , which form a cycle edge orbit of  $E$  in  $G_T$  with vertices  $v_1, v_{\alpha-1}$  of opposite parity. If  $|\bar{e}| \geq t+1$  then, as the edge  $e_{t+1}$  also has endpoints on  $v_1 \cup v_{\alpha-1}$ , it must lie in  $T$  in the annular region between the cycle formed by  $e_1, e_{\alpha-1}$  and some other cycle of  $E$ , which implies that  $e_{t+1}$  is parallel to  $e_1$  or  $e_{\alpha-1}$  in  $T$ , contradicting Lemma 2.1(b) (see Fig. 2(b)). Therefore  $|\bar{e}| \leq t$ .

If  $|\bar{e}| = t$  then every vertex  $v$  of  $G_T$  belongs to a unique cycle edge orbit  $c(v)$  of  $\bar{e}$ . Suppose that the vertices in  $c(v)$  are  $v$  and  $v'$ . Then it is not hard to see from Fig. 2(b) that  $v$  can have at most two incident positive nonloop edges of  $\bar{G}_T$  on each side of the cycle  $c(v)$ ; so if  $v$  has at least three incident positive nonloop edges of  $\bar{G}_T$ , then  $v'$  can have at most one incident positive nonloop edge in  $\bar{G}_T$  (see Fig. 2(b)).  $\square$

**3.2. Negative edges I.** The following fact is the starting point for our analysis of the size of the negative edges in  $\bar{G}_S$ ; its proof follows from [10, Lemma 2.8(2)], and we include it for the convenience of the reader.

**Lemma 3.2.** *Suppose  $(M, T_0)$  is not cabled. If  $t \geq 1$  and  $\bar{e}$  is a negative edge of  $\bar{G}_S$  with  $|\bar{e}| \geq t+1$ , then  $T$  is polarized and any subcollection of  $t$  consecutive edges in  $\bar{e}$  has exactly one edge orbit. In particular, all disk faces of  $G_S$  are even sided.*

*Proof.* Suppose  $t \geq 1$  and there is a negative edge  $\bar{e}$  in  $\bar{G}_S$  of size  $|\bar{e}| \geq t+1$ , with one endpoint in  $u_i$  and the other in  $u_{i'}$ . We may assume  $e_1, \dots, e_t, e_{t+1}, \dots$  are all the edges in  $\bar{e}$ , as shown in Fig. 3, oriented from  $u_i$  to  $u_{i'}$ .

The collections of edges  $E = \{e_1, \dots, e_t\}$  and  $E' = \{e_2, \dots, e_{t+1}\}$  induce the same permutation  $\sigma$ , of the form  $\sigma(x) = x + \alpha$  for some  $0 \leq \alpha < t$  (cf § 2.2), and  $\sigma$  has  $n = \gcd(t, \alpha)$  orbits. By Lemma 2.1(b), in  $G_T$ , the edge orbits of each collection  $E, E'$  are nontrivial disjoint cycles and the edges  $e_1$  and  $e_{t+1}$  are not parallel. Let  $\gamma, \gamma'$  be the edge orbits of  $E, E'$ , that contain the edges  $e_1, e_{t+1}$ , respectively. If  $n \geq 2$  then the edge  $e_{t+1}$  is necessarily located in between two distinct edge orbits of  $E$ , with both endpoints on the



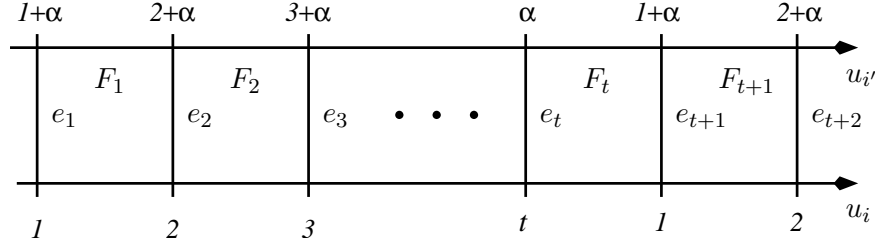


FIGURE 3.

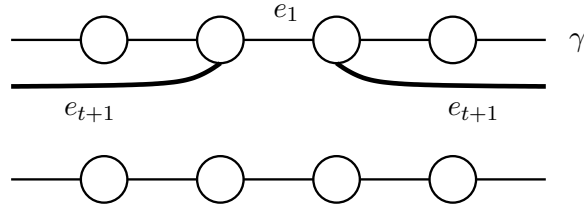


FIGURE 4.

same side of the cycle  $\gamma$  in  $G_T$ , as shown in Fig. 4. As the edges of  $\gamma'$  coincide with those of  $\gamma$ , except for the edge  $e_1$  which gets replaced by  $e_{t+1}$ , it follows that the cycle  $\gamma'$  bounds a disk in  $\hat{T}$ , contradicting Lemma 2.1(b). Therefore  $n = 1$ , so  $\sigma$ , and hence  $E$ , have a single orbit, and so  $T$  is polarized; thus, by the parity rule, all edges in  $G_S$  are negative, from which it follows that any boundary component of any face of  $G_S$  has an even number of sides.  $\square$

**3.3. Negative edges II: a construction.** For any properly embedded surface  $\mathcal{F}$  in a 3-manifold  $\mathcal{M}$ , we will denote by  $\mathcal{M}_{\mathcal{F}} = \mathcal{M} \setminus \text{int } N(\mathcal{F})$  the manifold obtained by cutting  $\mathcal{M}$  along  $\mathcal{F}$ ; if  $\mathcal{F}$  is orientable then  $N(\mathcal{F}) = \mathcal{F} \times I$ , where  $I = [0, 1]$ , and  $\partial \mathcal{M}_{\mathcal{F}}$  contains two copies  $\mathcal{F}^0 = \mathcal{F} \times 0$ ,  $\mathcal{F}^1 = \mathcal{F} \times 1$  of  $\mathcal{F}$ .

Given any collection  $E$  of mutually parallel, consecutive negative edges of  $G_S$  with  $|E| \geq 2$ , we define  $M_{T,E} \subset M$  as the submanifold obtained by cutting  $M$  along the union of  $T$  and the bigon faces cobounded by the edges of  $E$  in  $G_S$ . In this section we will take a closer look at the manifolds  $M_{T,E}$  constructed with large enough collections  $E$ . Observe  $M_T$  and  $M_{T,E}$  are irreducible manifolds.

Let  $t \geq 1$  and  $E = \{e_1, e_2, \dots, e_{t+1}\}$  be any collection of  $t + 1$  mutually parallel, consecutive, negative edges in  $G_S$ , running and oriented from the vertex  $u_i$  to the vertex  $u_{i'}$  of  $G_S$ , and labeled as in Fig. 3. By Lemma 3.3,  $T$  is polarized, hence nonseparating in  $M$ , so the permutation induced by  $E$  is of the form  $x \mapsto x + \alpha \pmod t$  with  $\gcd(t, \alpha) = 1$ . In what follows, for clarity, our figures will sometimes be sketched to represent scenarios for large  $t$ , but the arguments and constructions can be seen to hold for all  $t \geq 1$ .

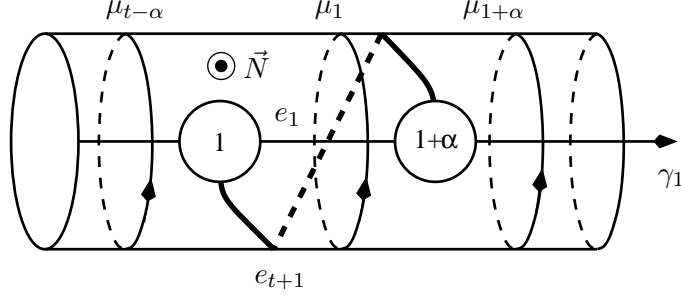


FIGURE 5.

It follows from the proof of Lemma 3.2 that the union of the edges in  $E$  form a subgraph of  $G_T$  isomorphic to the graph shown in Fig. 5, where  $e_1, \dots, e_t$  are represented by the horizontal edges and  $e_{t+1}$  by the thicker edge. Moreover, if  $\gamma_1, \gamma_2$  are the cycle edge orbits of the collections  $\{e_1, \dots, e_t\}$  and  $\{e_2, \dots, e_{t+1}\}$ , respectively, then  $\gamma_1$  is the oriented cycle comprised of all the horizontal edges in Fig. 5, while  $\gamma_2$  is obtained from  $\gamma_1$  by exchanging the edge  $e_1$  with the edge  $e_{t+1}$ . Hence  $\Delta(\gamma_1, \gamma_2) = 1$  in  $\widehat{T}$ . The situation gets somewhat simplified in the case  $t = 1$  from what is shown in Fig. 5, which deals with the cases  $t \geq 2$ .

Fig. 5 also shows a collection of  $t$  oriented circles  $\mu_i$ ,  $1 \leq i \leq t$ , each having the same slope in  $\widehat{T}$  as the cycle  $e_1 \cup e_{t+1}$  and labeled by the vertex along  $\gamma_1$  that precedes (following the orientation of  $\gamma_1$ ). Notice that the cycles  $\gamma_1$  and  $\gamma_2$  can be obtained from each other via one full Dehn twist on  $T$  along  $\mu_1$ .

Each vertex  $v_k$ , edge  $e_k$ , and circle  $\mu_k$  in  $T$  splits into two copies  $v_k^1, e_k^1, \mu_k^1 \subset T^1$ ,  $v_k^2, e_k^2, \mu_k^2 \subset T^2$ , with  $v_k^1, v_k^2$ ,  $e_k^1, e_k^2$ , and  $\mu_k^1, \mu_k^2$  parallel in  $N(T) = T \times I$  to  $v_k, e_k, \mu_k$ , respectively; Fig. 6 shows such parallelism for  $e_k, e_k^1, e_k^2$ .

Let  $\psi : T^1 \rightarrow T^2$  be the gluing homeomorphism that produces  $M$  out of  $M_T$ . We will orient  $e_k^1, e_k^2$  and  $\mu_k^1, \mu_k^2$  in the same direction as  $e_k, \mu_k$ , respectively, via the parallelism  $N(T) = T \times I$ , so that  $\psi(e_k^1) = e_k^2$  and  $\psi(\mu_k^1) = \mu_k^2$ , preserving orientations.

The edges  $e_1^1, e_2^1, \dots, e_t^1$  form a cycle in  $T^1$  which is parallel in  $N(T)$  to the cycle  $\gamma_1$ , while  $e_2^2, e_3^2, \dots, e_{t+1}^2$  form a cycle in  $T^2$  parallel in  $N(T)$  to  $\gamma_2$ . We will denote these cycles by  $\gamma_1^1 \subset T^1$  and  $\gamma_2^2 \subset T^2$ , respectively. Thus, while the cycles  $\gamma_1$  and  $\gamma_2$  intersect in  $T$ , the cutting process along  $T$  ‘separates’ them into the disjoint cycles  $\gamma_1^1, \gamma_2^2$ , with  $\psi(\gamma_1^1) = \gamma_2^2$ .

For each string  $I_{k,k+1}$  of  $T$ , we will call the annulus  $I'_{k,k+1} = I_{k,k+1} \cap M_T \subset \partial M_T$  a *string of  $M_T$* . Observe that the union of  $T^1, T^2$ , and the strings of  $M_T$  is one of the boundary components of  $M_T$ , of genus  $t + 1$ .

Consider now the bigons  $F_1, F_2, \dots, F_t$  of  $G_S$  cobounded by the edges of  $E$ , as shown in Fig. 3. We call the disks  $F'_k = F_k \cap M_T \subset M_T$ ,  $1 \leq k \leq t$ , the *faces of  $E$  in  $M_T$* ; these faces have corners along the strings  $I'_{k,k+1}$  and are

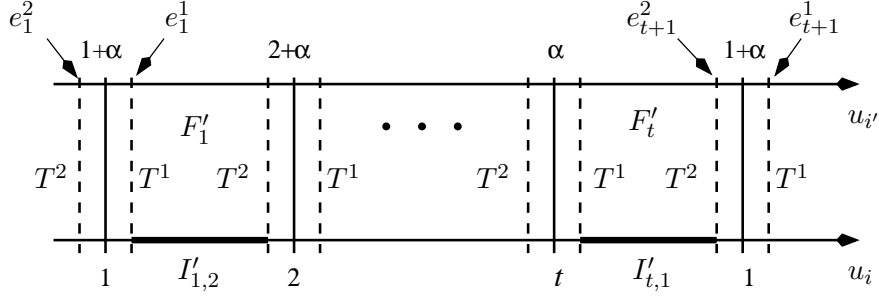


FIGURE 6.

properly embedded in  $M_T$ . For  $1 \leq k \leq t$ ,  $\partial F'_k$  consists of four segments: one corner along the string  $I'_{k,k+1}$ , one corner along  $I'_{k+\alpha,k+\alpha+1}$ , and the two edges  $e^1_k \subset T^1$  and  $e^2_{k+1} \subset T^2$  (see Fig. 6). Since  $|E| = t + 1$ , along each vertex  $u_i, u_{i'}$  each string of  $T$  appears exactly once among the corners of the bigon faces  $F_k$ . Thus, each string  $I'_{k,k+1}$  of  $M_T$  has exactly two corners coming from all the faces  $F_j$  in  $E$ , and these two corners cut  $I'_{k,k+1}$  into two rectangular pieces, which we denote by  $J_{k,k+1}, L_{k,k+1}$ .

It follows that the faces  $F'_k$  are embedded in  $M_T$  as shown in Fig. 7(a). To determine the location of the edges  $e^2_1 \subset T^2$  and  $e^1_{t+1} \subset T^1$ , consider the normal vector  $\vec{N}$  to  $T$  indicated in Fig. 5 by the tip of an arrow  $\odot$ , and orient  $T^1, T^2$  via normal vectors  $\vec{N}^1, \vec{N}^2$ , respectively, such that  $\vec{N}^1 = \vec{N}^2 = \vec{N}$  after identifying  $T^1$  with  $T^2$ ; these vectors are indicated in Fig. 7(a), and we will use them to identify the right hand and left hand sides of the cycles  $\gamma \subset T, \gamma^1_1 \subset T^1, \gamma^2_2 \subset T^2$  consistently. Since the oriented edge  $e_{t+1}$  has initial and terminal endpoints on the right and left hand sides of the oriented cycle  $\gamma$ , respectively, the endpoints of the edge  $e^i_{t+1} \subset T^i$  must behave the same way relative to the oriented cycle of edges  $e^1_1 \cup e^2_2 \cup \dots \cup e^i_t \subset T^i$  for  $i = 1, 2$ . Therefore the edges  $e^2_1 \subset T^2, e^1_{t+1} \subset T^1$  must be embedded as shown in Fig. 7(a) (up to Dehn twists in the annuli  $T^i \setminus \gamma^i_i$ ), and hence  $\mu^1_1, \mu^2_1$  must then be embedded in  $T^1, T^2$  as shown in Fig. 7(a).

Cutting  $M_T$  along the faces  $F'_k$  produces the irreducible submanifold  $M_{T,E} \subset M$ , which has a distinguished torus boundary component  $R_E$  that contains all the rectangles  $J_{k,k+1}, L_{k,k+1}$  and two copies of each face  $F'_k$ . The union of all these pieces forms two disjoint nontrivial annuli  $A_E, A'_E \subset R_E$ ; relabeling if necessary, we may assume that  $A_E$  contains all the rectangles  $J_{k,k+1}$ , while  $A'_E$  contains the  $L_{k,k+1}$ 's (see Fig. 7(b)).

So, if  $\widehat{M}_T$  is the manifold obtained by cutting  $M(\partial T)$  along  $\widehat{T}$ , it is not hard to see that  $\widehat{M}_T$  can be obtained from  $M_{T,E}$  by identifying  $A_E$  with  $A'_E$  in such a way that all pairs of rectangles  $J_{k,k+1}$  and  $L_{k,k+1}$  match.

A first approximation to the structures of  $M, M_T$ , and  $M(\partial T)$  is given in our next result.

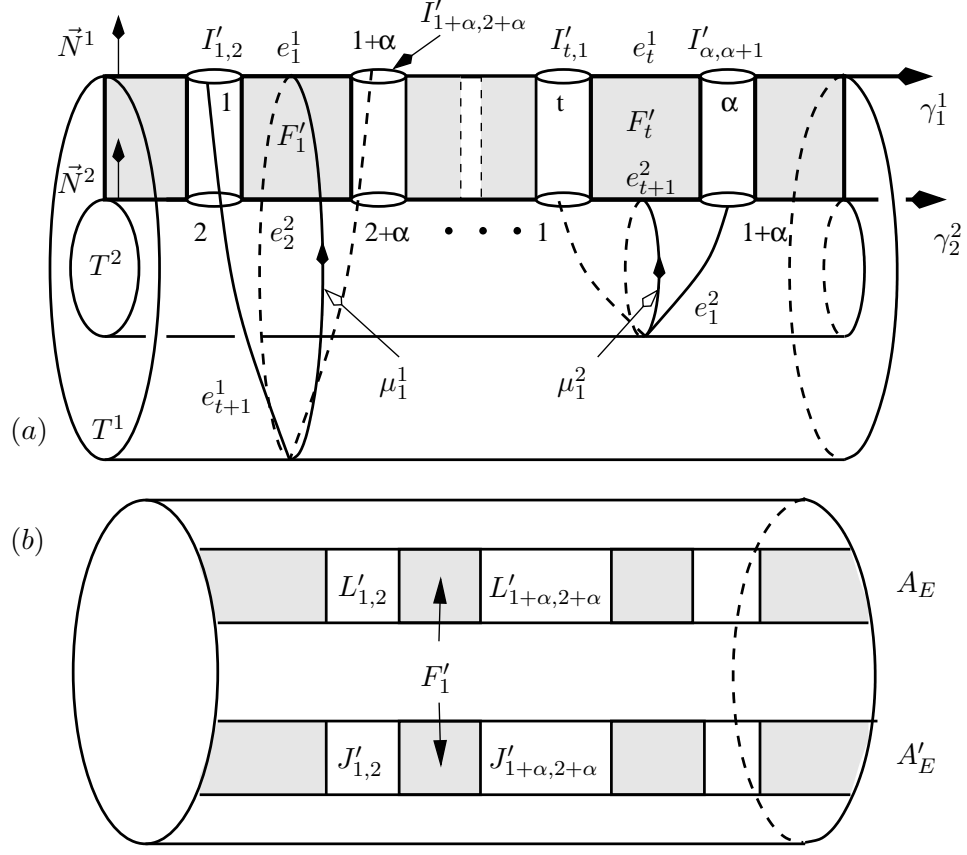


FIGURE 7.

**Lemma 3.3.** *Suppose that  $t \geq 1$  and  $\bar{e}$  is a negative edge of  $\bar{G}_S$  of size  $|\bar{e}| \geq t + 1$ .*

- (a) *If  $T$  is incompressible in  $M$ ,  $|\bar{e}| = t + 1$ , and the torus  $R_{\bar{e}} \subset \partial M_{T, \bar{e}}$  compresses in  $M_{T, \bar{e}}$ , then  $\partial M = T_0$  and  $\widehat{M}_T$  is a Seifert fibered space over the annulus with at most one singular fiber;*
- (b) *if  $|\bar{e}| \geq t + 2$  then  $M_T \approx T \times I$ , so  $\partial M = T_0$ ,  $T$  is incompressible in  $M$ , and  $M(\partial T)$  is an irreducible torus bundle over the circle with fiber  $\widehat{T}$ .*

*Proof.* For part (a), let  $c$  be the core of the annulus  $A_{\bar{e}} \subset R_{\bar{e}}$ . If  $D$  is a compression disk for  $R_{\bar{e}}$  in  $M_{T, \bar{e}}$  then, as  $T$  is incompressible and  $M_{T, \bar{e}}$  is irreducible, we must have  $d = \Delta(\partial D, c) \geq 1$  and  $M_{T, \bar{e}}$  a solid torus. Hence  $\partial M = T_0$  and  $\widehat{M}_T$  is a Seifert fibered space over the annulus with at most one singular fiber, of index  $d$ .

For part (b), let  $\{e_1, e_2, \dots, e_{t+1}, e_{t+2}\}$  be a collection of  $t + 2$  consecutive edges in  $\bar{e}$ , with edges and bigons labeled as in Fig. 3, and consider the manifold  $M_{T, E}$  corresponding to the family of edges  $E = \{e_1, e_2, \dots, e_{t+1}\}$ .

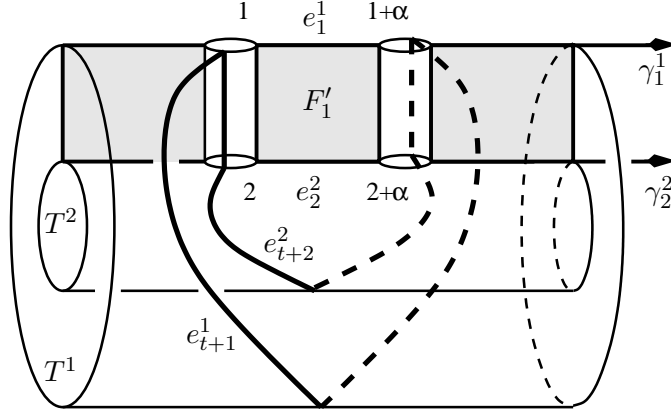


FIGURE 8.

As the edge  $e_{t+1}^1$  is not parallel in  $T^1$  to any of the edges of the cycle  $\gamma_1^1 \subset T^1$  (see Fig. 7(a)), the disk face  $F'_{t+1} = F_{t+1} \cap M_T$  is not parallel in  $M_T$  to any of the disks  $F'_k = F_k \cap M_T$  for  $1 \leq k \leq t$ , and hence  $F'_{t+1}$  is necessarily embedded in  $M_T$  as shown in Fig. 8 (with  $\partial F'_{t+1}$  the union of the thicker edges  $e_{t+1}^1, e_{t+2}^2$  and corners). It follows that  $F'_{t+1}$ , which also lies in  $M_{T,E}$ , intersects each annulus  $A_E, A'_E$  transversely in one spanning arc. Therefore, by the argument of part (a),  $\widehat{M}_T$  is a Seifert fibered space over the annulus with no singular fibers, so  $\widehat{M}_T \approx \widehat{T} \times I$ , from which it follows that  $M_T \approx T \times I$  and  $M(\partial T)$  is an irreducible torus bundle over the circle with incompressible fiber  $\widehat{T}$ .  $\square$

**3.4. Negative edges III.** In this section we will further assume that  $G_S$  has at least  $t + 2$  mutually parallel negative edges, no two of which are parallel in  $T$ , and determine the structure of  $M$  under these conditions.

We will use the following definitions. Let  $P$  be an oriented pair of pants with boundary components  $\mu_0, \mu_1, \mu_2$ , each given the induced orientation from  $P$ , and consider the manifold  $P \times S^1$ , where  $P$  is identified with some fixed copy  $P \times \{*\}$  in  $P \times S^1$ . We orient the manifold  $P \times S^1$  via a product orientation, so that the circles  $\{*\} \times S^1$  all follow the direction of an orientation normal vector of  $P$  in  $P \times S^1$ ; the boundary tori components  $T_i = \mu_i \times S^1$ ,  $i = 0, 1, 2$ , are then oriented by an outside pointing normal vector  $\vec{N}_i$ .

Let  $\phi : T_1 \rightarrow T_2$  be an orientation reversing homeomorphism such that

$$(1) \quad \phi(\mu_1) = -\mu_2,$$

where  $-\mu_2$  is the circle  $\mu_2$  with the opposite orientation. Then the quotient manifold  $P \times S^1 / \phi$  is orientable, irreducible, and has incompressible boundary the torus  $T_0$ . Also, under the quotient map  $P \times S^1 \rightarrow P \times S^1 / \phi$ ,  $P$  gives rise to a once punctured torus  $T_P$  in  $P \times S^1 / \phi$  with boundary slope  $\mu_0$ , and the tori  $T_1, T_2$  give rise a closed, nonseparating, incompressible torus

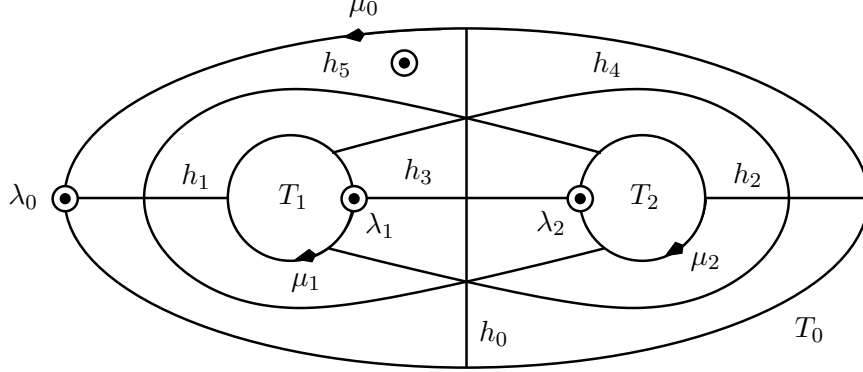


FIGURE 9.

$T'_P \subset P \times S^1/\phi$  which intersects  $T_P$  transversely in a single circle corresponding to  $\mu_1 = -\mu_2$ .

Consider the arcs  $h_i \subset P$ ,  $0 \leq i \leq 5$ , shown in Fig. 9; these arcs give rise to essential annuli  $h_i \times S^1 \subset P \times S^1$ , which are the unique (up to isotopy) properly embedded essential annuli in  $P \times S^1$ ; in fact, the annulus  $h_0 \times S^1$  is the unique essential surface in  $P \times S^1$  with boundary on  $T_0$  (a similar statement holds for  $h_4 \times S^1$  and  $h_5 \times S^1$ ). In particular, the pair  $(P \times S^1, T_0)$  is not cabled. The boundary components of these annuli correspond to three slopes  $\lambda_i \subset T_i$ ,  $i = 0, 1, 2$ ; these are the unique slopes on the  $T_i$ 's arising from any Seifert fibration on  $P \times S^1$ . We will orient all the circles  $\mu_i, \lambda_i$ ,  $i = 0, 1, 2$ , as shown in Fig. 9, where the tips of arrows  $\odot$  indicate directions of tangent/normal vectors, and each circle  $\mu_i$  is also labeled by the torus  $T_i$  that contains it; thus, the  $\lambda_i$ 's have the same orientation as the fibers  $\{x\} \times S^1$ . The first homology group  $H_1(P \times S^1)$  (with integer coefficients) is then freely generated by  $\mu_1, \mu_2, \lambda_0$ , and the following relations hold:

$$(2) \quad \lambda_0 = \lambda_1 = \lambda_2, \quad \mu_0 + \mu_1 + \mu_2 = 0.$$

The above orientation scheme allows us to define intersection numbers between two oriented circles  $c, c'$  in any boundary torus  $T_i$  of  $P \times S^1$ , by requiring that  $c \cdot c'$  be positive at a point  $x \in c \cap c'$  of transverse intersection iff the tangent vectors  $\vec{v}, \vec{v}'$  to  $c, c'$  at  $x$ , respectively, yield an orientation triple  $(\vec{v}, \vec{v}', \vec{N}_i)$  of  $P \times S^1$  at  $x$ . With this convention, the fact that  $\phi$  is orientation reversing can be restated as follows:

$$(3) \quad \text{for any two oriented circles } c, c' \subset T_1, \quad \phi(c) \cdot \phi(c') = -c \cdot c'.$$

Notice that  $\mu_1 \cdot \lambda_1 = +1$  and  $\mu_2 \cdot \lambda_2 = +1$ . Relative to these orientation frames  $\mu_1, \lambda_1$  of  $H_1(T_1)$  and  $\mu_2, \lambda_2$  of  $H_1(T_2)$ , we can write  $\phi(\lambda_1) = m\mu_2 + r\lambda_2$  in  $H_1(T_2)$  for some relatively prime integers  $m, r$ , and then it follows from (1), (3), and  $\mu_1 \cdot \lambda_1 = +1 = \mu_2 \cdot \lambda_2$  that  $r = +1$ , so

$$(4) \quad \phi(\lambda_1) = m\mu_2 + \lambda_2 \quad \text{in } H_1(T_2).$$

The homeomorphism  $\phi$  is determined up to isotopy by its action on first homology, which, relative to the orientation frames  $\mu_1, \lambda_1$  and  $\mu_2, \lambda_2$ , is given by the matrix  $[\phi] = \begin{pmatrix} -1 & m \\ 0 & 1 \end{pmatrix}$ ; so we will also denote the manifold  $P \times S^1/\phi$  by  $P \times S^1/[m]$ , and use the notation

$$(P \times S^1/[m], T_P, T'_P, T_0, \mu_0, \lambda_0)$$

to stress the presence of the specific objects  $T_P, T'_P, T_0, \mu_0, \lambda_0 \subset P \times S^1/\phi$  constructed above. Since the quotient manifolds  $P \times S^1/\phi$  and  $P \times S^1/\phi^{-1}$  are homeomorphic, and  $[\phi^{-1}] = \begin{pmatrix} -1 & -m \\ 0 & 1 \end{pmatrix}$ , switching the roles of  $T_1, T_2$  in  $P \times S^1/\phi^{-1}$  gives rise to a homeomorphism  $P \times S^1/[m] \approx P \times S^1/[-m]$ , and so we may assume that  $m \geq 0$ . Finally, we identify the cut manifold  $(P \times S^1/[m])_{T'_P}$  with  $P \times S^1$ .

The main result of this section can now be stated as follows:

**Proposition 3.4.** *Let  $(T, \partial T) \subset (M, T_0)$  be a  $\mathcal{K}$ -incompressible torus and  $S \subset M$  a surface which intersects  $T$  in essential graphs  $G_S, G_T$ . Set  $t = |\partial T| \geq 1$ , and suppose that  $G_S$  has at least  $t + 2$  mutually parallel, consecutive negative edges, no two of which are parallel in  $T$ . If  $t = 1$  then  $M$  is the exterior of the trefoil knot, while if  $t \geq 2$  then  $M = (P \times S^1/[m], T_P, T'_P, T_0, \mu_0, \lambda_0)$  with  $T_P$  having the same boundary slope  $\mu_0$  as  $T$ , and the following hold:*

- (a)  $(M, T_0)$  is not cabled;
- (b)  $M(\alpha)$  is irreducible and toroidal for any slope  $\alpha \neq \lambda_0$ , and  $M(\lambda_0) \approx S^1 \times S^2 \# L$  for some closed 3-manifold  $L$  of genus  $\leq 1$ ;
- (c)  $M = P \times S^1/[m]$  contains a punctured  $\mathcal{K}$ -incompressible torus with boundary slope  $\alpha \neq \mu_0$  iff  $m = 1, 2, 4$  and  $\alpha$  is the slope of  $\mu_0 - (4/m)\lambda_0$ ; in such case,  $\Delta(\alpha, \mu_0) = 4/m = 1, 2, 4$  and  $M$  also contains an essential  $q$ -punctured Klein bottle of boundary slope  $\alpha$ , where  $(m, q) = (1, 1), (2, 1)$ , or  $(4, 2)$ .  $\square$

Proposition 3.4 follows immediately from Lemmas 3.5, 3.6, 3.7, and 3.8 below. In what follows we will also use the notation of §3.3; as usual, we may draw some figures as if  $t$  were large only for clarity.

**Lemma 3.5.** *If  $t = 1$  then  $M$  is the exterior of the trefoil knot.*

*Proof.* Let  $e_1, e_2, e_3$  be three distinct mutually parallel, consecutive edges in  $G_S$  which are not parallel in  $G_T$ , running from  $u_i$  to  $u_{i'}$ , and let  $F_1, F_2$  be the bigon disk faces they cobound in  $G_S$ , as shown in Fig. 10(a). The graph  $\overline{G}_T$  is then isomorphic to the graph shown in Fig. 18(a).

As in the proof of Lemma 3.3, the edges  $e_1^1, e_2^2$  and  $e_2^1, e_3^2$  must lie in  $T^1, T^2$  as shown in Fig. 10(c), cobounding the faces  $F'_1, F'_2$  of  $M_T$ , respectively. To locate the edges  $e_1^2 \subset T^2$  and  $e_3^1 \subset T^1$ , first observe that locally the edges  $e_1, e_2, e_3$  produce the pattern around the vertex  $v_1$  of  $G_T$  shown in Fig. 10(b), say with the edges  $e_1, e_2, e_3$  repeating cyclically twice around  $v_1$ , in that order, so that exactly the same pattern must be present around each copy  $v_1^1 \subset T^1, v_1^2 \subset T^2$  of  $v_1$ . Therefore, if  $e_3^1$  is embedded in  $T^1$  as shown

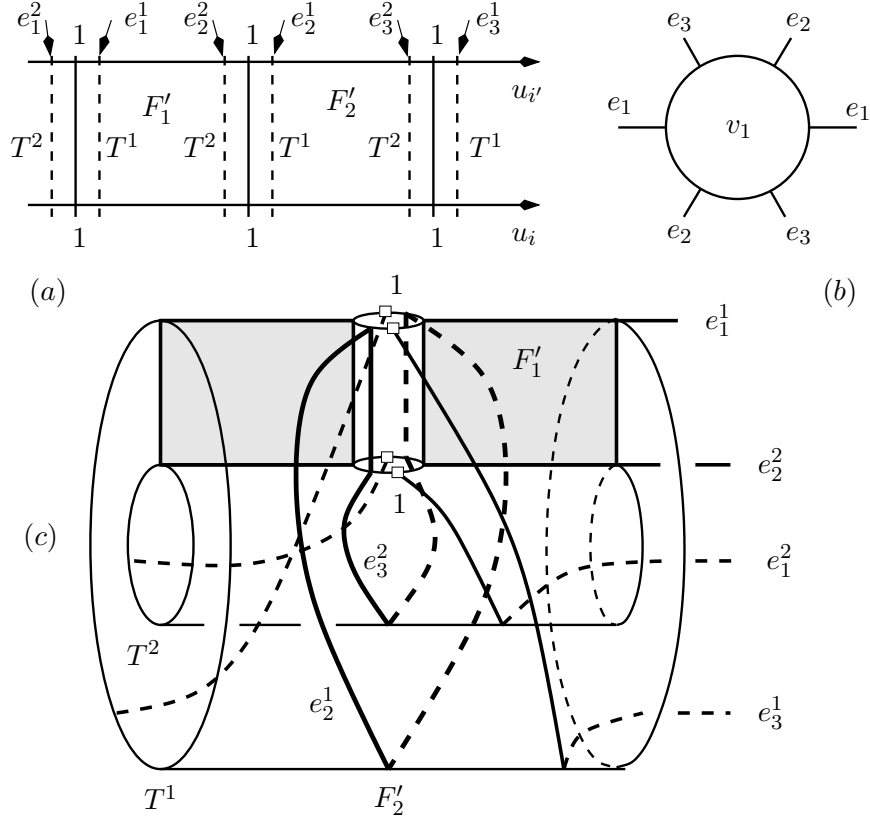


FIGURE 10.

in Fig. 10(c), then  $e_1^2$  must be embedded in  $T^2$  as shown in Fig. 10(c), and so these two edges  $e_3^1, e_1^2$ , along with two spanning arcs on the string  $I'_{1,1}$  of  $M_T$ , cobound a rectangular disk  $D$  in  $M_T$  disjoint from  $F'_1 \cup F'_2$  (but not necessarily from  $S$ ).

Let  $B \subset M$  be the surface obtained from the union of  $F'_1, F'_2, D$ , after identifying  $T^1$  with  $T^2$  in  $M_T$  via  $\psi$  so that  $e_k^1 = e_k^2$  for  $k = 1, 2, 3$ . Then  $B$  is either an annulus or a Moebius band in  $(M, T_0)$  which intersects  $T$  in essential graphs consisting of exactly 3 edges, so that the graph  $G_{T,B} = B \cap T \subset T$  has two triangle faces  $C_1, C_2$  (see Fig. 18(a)).

If  $B$  is an annulus then  $\Delta(\partial B, \partial T) = 3$ ,  $B$  is neutral by the parity rule, and the faces  $C_1$  and  $C_2$  locally lie on opposite sides of  $B$ . Hence cutting the irreducible manifold  $M$  along  $B$  produces two solid tori  $V_1, V_2$  with corresponding meridian disks  $C_1, C_2$ . Since all edges of the graph  $G_{T,B}$  are positive, each disk  $C_1, C_2$  intersects the annulus  $B$  coherently and transversely in 3 spanning arcs, and so  $M = V_1 \cup_B V_2$  is homeomorphic to a Seifert fibered space with base a disk and two singular fibers of indices 3, 3. However,  $T$  is the union of the two meridian disks  $C_1$  and  $C_2$  along the 3 edges of  $B \cap T$ , and it is not hard to see that any such union produces a pair



of pants in  $M$ , not a once punctured torus. Therefore  $B$  must be a Moebius band, with  $\Delta(\partial B, \partial T) = 6$ , and cutting  $M$  along  $B$  produces a solid torus with meridian disk either triangle face  $C_1$  or  $C_2$  of  $G_{T,B}$ . Since all edges of the graph  $G_{T,B}$  are positive,  $M$  must be homeomorphic to a Seifert fibered space with base a disk and two singular fibers of indices 2, 3, which is the trefoil knot exterior.  $\square$

The rest of this section is devoted to the cases  $t \geq 2$ .

**Lemma 3.6.** *If  $t \geq 2$  then  $M = (P \times S^1/[m], T_P, T'_P, \mu_0, \lambda_0)$  for some integer  $m$ , with  $T_P$  and  $T$  having the same boundary slope  $\mu_0$ .*

*Proof.* Suppose that  $t \geq 2$  and  $\bar{e}$  is a negative edge of  $\overline{G}_S$  with  $|\bar{e}| \geq t + 2$ . We assume that  $e_1, \dots, e_t, e_{t+1}, e_{t+2}, \dots$  are all the edges in  $\bar{e}$ , labeled as in Fig. 3, and oriented from  $u_i$  to  $u_{i'}$ .

Let  $\psi : T^1 \rightarrow T^2$  be the gluing map that produces  $M$  out of  $M_T$ . As in the proof of Lemma 3.3, the face  $F'_{t+1} = F_{t+1} \cap M_T$  is properly embedded in  $M_T$  with boundary as shown in Fig. 8, and  $M_T = T \times I$ .

Consider now the oriented circles  $\mu_1, \dots, \mu_t$  embedded in  $T$  as shown in Fig. 5. Recall each circle  $\mu_k$  is labeled by the vertex of  $T$  that precedes it along the oriented cycle  $\gamma_1$  generated by  $e_1, \dots, e_t$  in  $T$ , and that  $\mu_k$  splits into two copies  $\mu_k^1 \subset T^1$  and  $\mu_k^2 \subset T^2$ , which are oriented in the same direction as  $\mu_k$  within  $N(T)$ ; thus, all circles  $\mu_k^1, \mu_k^2$  are coherently oriented in  $T^1, T^2$ . From Figs. 7(a) and 8 and the fact that  $e_1^2, e_{t+2}^2$  are disjoint and nonparallel in  $T^2$ , it follows that all circles  $\mu_k^1 \subset T^1$  and  $\mu_k^2 \subset T^2$  are embedded as shown in Fig. 11.

Therefore, the faces  $F'_1$  and  $F'_{t+1}$  can be isotoped in  $M_T$  to construct an annulus  $A_1 \subset M_T$  with boundary the circles  $\mu_1^1 \cup \mu_2^2$ , which under their given orientations remain coherently oriented relative to  $A_1$ . Via the product structure  $M_T = T \times I$ , it is not hard to see that each pair of circles  $\mu_k^1, \mu_{k+1}^2$  cobounds such an annulus  $A_k \subset M_T$  for  $1 \leq k \leq t$ , with the oriented circles  $\mu_k^1, \mu_{k+1}^2$  coherently oriented relative to  $A_k$ ; these annuli  $A_k$  can be taken to be mutually disjoint and  $I$ -fibered in  $M_T = T \times I$ . Since  $\psi(\mu_k^1) = \mu_k^2$  (preserving orientations), the union  $A_1 \cup A_2 \cdots \cup A_t$  yields a closed nonseparating torus  $T''$  in  $M$ , on which the circles  $\mu_1, \mu_2, \dots, \mu_t$  appear consecutively, coherently oriented, and in this order.

Thus, the region in  $M_T$  cobounded by any pair  $A_i, A_j$  of consecutive annuli has a product structure of the form  $P_{i,j} \times I$ , where  $P_{i,j}$  is the pants region cobounded by the boundary circles of  $A_i, A_j$  in  $T^1$ ; since  $M_{T''}$  is the union of these regions  $P_{i,j} \times I$ , glued along their pant boundary pieces  $P_{i,j} \times 0, P_{i,j} \times 1$  via the map  $\psi$ , it follows that  $M_{T''}$  has a product structure of the form  $P \times S^1$ , where  $P$  is any of the pants  $P_{i,j}$ .

Now, in  $M$ ,  $P$  has one boundary component  $\partial_0 P$  on  $T_0$ , of the same slope as  $\partial T$ , while the other two boundary components  $\partial_1 P, \partial_2 P$  lie on  $T''$  and are disjoint. From the point of view of  $T''$ , the circles  $\partial_1 P, \partial_2 P$  coincide with two of the circles  $\mu_k \subset T''$ , whose given orientations are coherent along

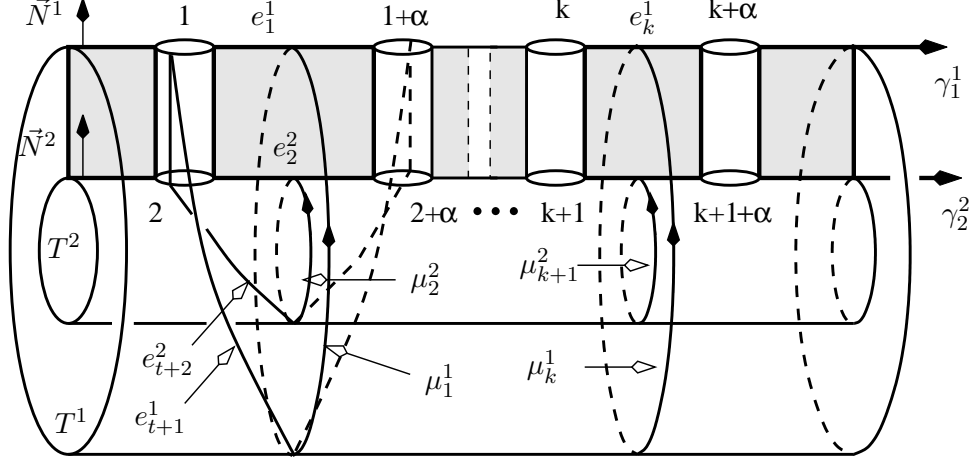


FIGURE 11.

$T''$ ; with such orientations, the circles  $\mu_k$  are then also coherently oriented relative to  $P$ . Therefore  $P$  can be isotoped in  $M$  so that  $\partial_1 P = \partial_2 P$  on  $T''$ , giving rise to a once punctured torus  $T_P$  in  $M$  of the same boundary slope as  $T$  which intersects  $T''$  in a circle of the same slope as the  $\mu_k$ 's.

It follows that  $M$  is a manifold of the form  $(P \times S^1/[m], T_P, T'_P, T_0, \mu_0, \lambda_0)$ , with  $T_P = T'$  and  $T'_P = T''$ .  $\square$

**Lemma 3.7.** *If  $M = (P \times S^1/[m], T_P, T'_P, T_0, \mu_0, \lambda_0)$  then  $(M, T_0)$  is not cabled,  $M(\lambda_0) \approx S^1 \times S^2 \# L$  for some closed 3-manifold  $L$  of genus  $\leq 1$ , and  $M(\alpha)$  is irreducible with  $T'_P$  incompressible in  $M(\alpha)$  for  $\alpha \neq \lambda_0$ .*

*Proof.* Write  $M = P \times S^1/\phi$  with  $\phi$  the gluing map  $\phi : T_1 \rightarrow T_2$ . Clearly, for any slope  $\alpha \neq \lambda_0$ ,  $(P \times S^1)(\alpha)$  is an irreducible Seifert fibered space over an annulus with at most one singular fiber. Therefore the tori  $T_1 \cup T_2 = \partial(P \times S^1)(\alpha)$  are incompressible in  $(P \times S^1)(\alpha)$ , so  $M(\alpha) = (P \times S^1)(\alpha)/\phi$  is irreducible and hence the nonseparating torus  $T'_P$  is incompressible in  $M(\alpha)$ .

Consider now the manifold  $M(\lambda_0)$ . Let  $A$  be the nonseparating and neutral annulus  $h_0 \times S^1 \subset P \times S^1$ , of boundary slope  $\lambda_0$ . Then  $\hat{A}$  is a nonseparating 2-sphere in  $M(\lambda_0)$  disjoint from the nonseparating torus  $T'_P$ . Observe that  $T'_P$  compresses in  $M(\lambda_0)$ , on both sides, via the disks generated by the annuli  $h_1 \times S^1$  and  $h_2 \times S^1$  of  $P \times S^1$ , whose boundaries are the circles  $\lambda_1 \subset T_1$  and  $\lambda_2 \subset T_2$ . Thus, cutting  $M(\lambda_0)$  along  $\hat{A} \cup T'_P$  yields two once punctured solid tori  $V_1, V_2$  with torus boundary components  $T_1, T_2$  and meridian disks of boundary slopes  $\lambda_1 \subset T_1, \lambda_2 \subset T_2$ , respectively. Gluing  $V_1$  to  $V_2$  along  $T_1, T_2$  via  $\phi$  then produces a twice punctured manifold  $L^-$ ; since  $|\phi(\lambda_1) \cdot \lambda_2| = m$ , identifying the two spherical boundary components of  $L^-$  via  $\phi$  produces the manifold  $M(\lambda_0) = S^1 \times S^2 \# L$ , where  $L = S^1 \times S^2, S^3$ , or a lens space for  $m = 0, 1$ , or  $m \geq 2$ , respectively.

Finally, suppose that  $(M, T_0)$  is cabled, with essential cabling annulus  $A'$ . Then  $A'$  is separating and hence neutral; as the annulus  $A$  is also neutral, by the parity rule  $A$  and  $A'$  must have the same boundary slope  $\lambda_0 \subset T_0$ . Since  $A$  is the unique essential surface in  $P \times S^1$  with boundary on  $T_0$ , it follows that, after isotoping  $A'$  in  $M$  so as to intersect  $T'_P$  transversely and minimally, we must have  $|A' \cap T'_P| > 0$ . Thus  $A' \cap P \times S^1$  is a collection of essential annular components, each of which must then be isotopic to some annulus  $h_i \times S^1 \subset P \times S^1$ ,  $1 \leq i \leq 5$ . It is not hard to see that two such components must be isotopic to  $h_1 \times S^1$  and  $h_2 \times S^1$ , which implies that  $\phi(\lambda_1) = \lambda_2$ , whence  $m = 0$  and so  $M(\lambda_0) = S^1 \times S^2 \# S^1 \times S^2$  by the argument above, contradicting the fact that  $A'$  being a cabling annulus implies that  $M(\partial A') = M(\lambda_0)$  has a lens space connected summand.  $\square$

**Lemma 3.8.**  *$M = P \times S^1/[m]$  contains a punctured  $\mathcal{K}$ -incompressible torus with boundary slope  $\alpha \neq \mu_0$  iff  $m = 1, 2, 4$  and  $\alpha$  is the slope of  $\mu_0 - (4/m)\lambda_0$ ; in such case,  $\Delta(\alpha, \mu_0) = 4/m = 1, 2, 4$  and  $M$  also contains an essential  $q$ -punctured Klein bottle of boundary slope  $\alpha$ , where  $(m, q) = (1, 1), (2, 1),$  or  $(4, 2)$ .*

*Proof.* Let  $R$  be a punctured essential torus or Klein bottle in  $M$ . Since the only connected essential surface in  $P \times S^1$  with boundary on  $T_0$  is the annulus  $h_0 \times S^1$ , after isotoping  $R$  in  $M$  so that it intersects  $T'_P$  transversely and minimally, we must have  $|R \cap T'_P| > 0$ , with each circle component of  $R \cap T'_P$  nontrivial in both  $R$  and  $T'_P$  and each component of  $R' = R \cap P \times S^1$  essential in  $P \times S^1$ . Isotoping  $R'$ , we may assume that  $R'$  and  $P$  intersect transversely in essential graphs.

**Claim 1.**  *$\alpha \neq \lambda_0$ , so  $M(\alpha)$  is irreducible and each component of  $R \cap T'_P$  is nontrivial in both  $\widehat{R}$  and  $T'_P$ .*

Clearly, there is some edge  $x$  in the essential graph  $R' \cap P \subset P$  for which at least one endpoint lies on  $T_0$ , ie,  $x$  is isotopic in  $P$  to  $h_0, h_1$ , or  $h_2$ . If  $\alpha = \lambda_0$  then, as  $R'$  is essential, the annulus  $A' = x \times S^1$  can be isotoped in  $P \times S^1$  so as to be disjoint from  $R'$ , which implies that  $R'$  lies in the cut manifold  $(P \times S^1)_{A'}$ , where it is incompressible. As  $(P \times S^1)_{A'}$  consists of one or two copies of  $(\text{torus}) \times I$ 's, it follows that  $R'$  must be a union of annuli, and hence that  $R$  is an annulus, which is not the case. Therefore  $\alpha \neq \lambda_0$  and hence, by Lemma 3.7,  $M(\alpha)$  is irreducible and the nonseparating torus  $T'_P$  is incompressible in  $M(\alpha)$ , whence each component of  $R \cap T'_P$  must be nontrivial in both  $\widehat{R}$  and  $T'_P$ .  $\square$ (Claim 1)

Thus, by Lemma 3.7 and Claim 1,  $(M, T_0)$  is not cabled and  $M(\alpha)$  is irreducible. Let  $Q$  be any component of  $R \cap P \times S^1$ ; by Claim 1,  $Q$  is an essential punctured annulus with two boundary components  $\partial_1 Q, \partial_2 Q$  in  $T_1 \cup T_2$ , and without loss of generality we may assume that  $q = |\partial Q \cap T_0| > 0$ . If  $\partial_1 Q \cup \partial_2 Q \subset T_1$  or  $\partial_1 Q \cup \partial_2 Q \subset T_2$  then  $Q$  boundary compresses in  $P \times S^1$  relative to  $T_0$  via the annulus  $h_2 \times S^1$  or  $h_1 \times S^1$ , respectively, contradicting

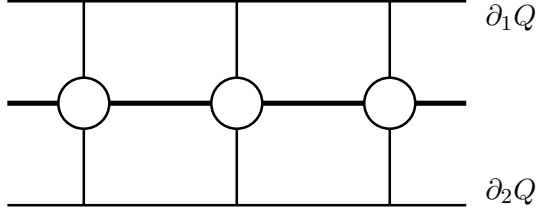


FIGURE 12.

the fact that  $R$  is essential in  $M$ ; thus we may assume that  $\partial_1 Q \subset T_1$  and  $\partial_2 Q \subset T_2$ .

**Claim 2.**  $\Delta(\alpha, \lambda_0) = 1$  and all components of  $\partial Q \cap T_0$  are coherently oriented in  $T_0$ .

Isotope  $Q$  so that it intersects the annuli  $(h_0 \cup h_1 \cup h_2) \times S^1 \subset P \times S^1$  transversely in essential graphs; then for  $i = 0, 1, 2$  each graph  $h_i \times S^1 \cap Q \subset h_i \times S^1$  consists of  $\Delta(\alpha, \lambda_0) \cdot q \geq q$  parallel edges, all of which are internal and negative for  $i = 0$ . The union of any  $q$  consecutive edges in the graphs  $h_i \times S^1 \cap Q \subset h_i \times S^1$  for  $i = 1, 2$  produce a subgraph in  $Q$  of the type shown in Fig. 12 (vertical thin edges). Therefore, any  $q$  consecutive edges of the graph  $h_0 \times S^1 \cap Q \subset h_0 \times S^1$  necessarily lie in  $Q$  like the thick horizontal edges shown in Fig. 12, so any edge of the graph  $h_0 \times S^1 \cap Q \subset h_0 \times S^1$  is parallel in  $Q$  to some horizontal edge of Fig. 12. Since the pair  $(P \times S^1, T_0)$  is not cabled, it follows from Lemma 2.1(b) that we must have  $\Delta(\alpha, \lambda_0) \cdot q = q$ , so  $\Delta(\alpha, \lambda_0) = 1$ , in which case the edges of the graph  $h_0 \times S^1 \cap Q \subset h_0 \times S^1$ , all of which are negative, form a single cycle in  $Q$ , which implies that all the components of  $\partial Q \cap T_0$  are coherently oriented in  $T_0$ .  $\square$ (Claim 2)

We now select the orientation on  $Q$  which induces the orientation on any component  $c$  of  $\partial Q \cap T_0$  such that  $c \cdot \lambda_0 > 0$ . Since  $\Delta(\alpha, \lambda_0) = 1$  by Claim 2, we can write  $\alpha = \mu_0 + b_0 \lambda_0$ , where  $|b_0| = \Delta(\alpha, \mu_0)$ , and so

$$\partial Q \cap T_0 = q\alpha = q(\mu_0 + b_0 \lambda_0) \text{ in } H_1(T_0),$$

$$\partial_1 Q = a_1 \mu_1 + b_1 \lambda_1 \text{ in } H_1(T_1), \quad \text{and}$$

$$\partial_2 Q = a_2 \mu_2 + b_2 \lambda_2 \text{ in } H_1(T_2),$$

for some pairs  $a_1, b_1$  and  $a_2, b_2$  of relatively prime integers. Since  $\partial Q = 0$  in  $H_1(P \times S^1)$ , by (2) we have that

$$0 = \partial Q \cap T_0 + \partial_1 Q + \partial_2 Q = (a_1 - q)\mu_1 + (a_2 - q)\mu_2 + (qb_0 + b_1 + b_2)\lambda_0,$$

and hence  $a_1 = a_2 = q$  and  $qb_0 = -(b_1 + b_2)$ , so that

$$(5) \quad \partial_1 Q = q\mu_1 + b_1 \lambda_1 \text{ in } H_1(T_1) \quad \text{and} \quad \partial_2 Q = q\mu_2 + b_2 \lambda_2 \text{ in } H_1(T_2).$$

Observe that  $\phi(\partial_1 Q) = \pm \partial_2 Q$  in  $H_1(T_2)$  since  $\phi$  maps the circle  $\partial_1 Q \subset T_1$  onto a circle in  $T_2$  of the same slope as  $\partial_2 Q$ .

**Claim 3.**  $\phi(\partial_1 Q) = +\partial_2 Q$  in  $H_1(T_2)$ ,  $(m, q) = (1, 1)$ ,  $(2, 1)$ , or  $(4, 2)$ , and  $\alpha = \mu_0 - (4/m)\lambda_0$ .

We have  $\phi(\partial_1 Q) = \varepsilon \partial_2 Q$  in  $H_1(T_2)$  for some  $\varepsilon \in \{\pm 1\}$ ; from (1), (4), and (5), it follows that

$$-q\mu_2 + b_1(m\mu_2 + \lambda_2) = \varepsilon(q\mu_2 + b_2\lambda_2) \text{ in } H_1(T_2),$$

and hence that  $b_1 m = (1 + \varepsilon)q$  and  $b_1 = \varepsilon b_2$ . If  $\varepsilon = -1$  then  $b_1 = -b_2$  and so  $qb_0 = -(b_1 + b_2) = 0$ ; but then  $b_0 = 0$ , whence  $\alpha = \mu_0$ , which is not the case. Hence  $\varepsilon = +1$ , so  $b_1 = b_2 = 2q/m$  and  $b_0 = -4/m$ , and so  $\Delta(\alpha, \mu_0) = |b_0| = 4/m \geq 1$ . In particular,  $m = 1, 2, 4$  and  $\alpha = \mu_0 - (4/m)\lambda_0$ , and since  $a_1 = q$  and  $b_1 = 2q/m$  are relatively prime integers we must have  $(m, q) = (1, 1)$ ,  $(2, 1)$ , or  $(4, 2)$ .  $\square$ (Claim 3)

Therefore, for each pair  $(m, q)$  listed in Claim 3,  $Q$  is  $q$ -punctured annulus which can be isotoped in  $P \times S^1$  so that  $\phi(\partial_1 Q) = \partial_2 Q$  in  $T_2$ , giving rise to a  $q$ -punctured Klein bottle  $Q'$  in  $M = P \times S^1/[m]$  with boundary slope  $\alpha = \mu_0 - (4/m)\lambda_0$ .

Now, since  $M(\alpha)$  is irreducible, the closed Klein bottle  $\widehat{Q}'$  is necessarily incompressible in  $M(\alpha)$ . So, if  $Q'$  is not essential in  $M$  then a compression or boundary compression of  $Q'$  gives rise to either a  $(q-1)$ -punctured Moebius band  $B$  in  $(M, T_0)$  or a closed Klein bottle  $R''$  in  $M$ . In the first case,  $\widehat{B}$  is a projective plane in the irreducible manifold  $M(\alpha)$ , which implies that  $M(\alpha)$  is homeomorphic to  $RP^3$ , contradicting the fact that  $M(\alpha)$  is a toroidal manifold for  $\alpha \neq \lambda_0$ . And in the second case, the closed Klein bottle  $R''$  must be incompressible in the irreducible manifold  $M$ , whence  $R''$  can be isotoped to intersect  $T'_P$  transversely and minimally, so that  $|R'' \cap T'_P| > 0$  and  $R'' \cap P \times S^1$  consists of annuli, all of which are essential in  $P \times S^1$ ; since any such annulus must then be isotopic to one of the annuli  $h_i \times S^1$ ,  $i = 3, 4, 5$ , it follows that  $m = 0$ , which is not the case. Therefore  $Q'$  is essential in  $M$ .

Conversely, let  $(m, q)$  be one of the pairs  $(1, 1)$ ,  $(2, 1)$ ,  $(4, 2)$ , and let  $\alpha = \mu_0 - (4/m)\lambda_0$ ; then a punctured annulus  $Q$  can be constructed in  $P \times S^1$  with  $q$  punctures in  $T_0$  of slope  $\alpha$  and one puncture in  $T_i$  of slope  $q\mu_i + (2q/m)\lambda_i$  for  $i = 1, 2$ , by homologically summing, in a suitable way,  $q$  copies of  $P$  and  $2q/m$  copies of each annulus  $h_1 \times S^1$ ,  $h_2 \times S^1$ . Since any homeomorphism  $\phi : T_1 \rightarrow T_2$  that homologically maps  $\mu_1$  onto  $-\mu_2$  and  $\lambda_1$  onto  $m\mu_2 + \lambda_2$  also maps  $q\mu_1 + (2q/m)\lambda_1$  onto  $q\mu_2 + (2q/m)\lambda_2$ , the lemma follows.  $\square$

#### 4. BOUNDARY SLOPES OF $\mathcal{K}$ -INCOMPRESSIBLE TORI

In this section we assume that  $(M, T_0)$  is not cabled and that  $(F_1, \partial F_1)$  and  $(F_2, \partial F_2)$  are  $\mathcal{K}$ -incompressible tori in  $(M, T_0)$  with boundary slopes at distance  $\Delta \geq 6$  and essential graphs of intersection; by Lemma 2.1(c), both Dehn filled manifolds  $M(r_1)$  and  $M(r_2)$  are irreducible,

We will use the generic notation  $\{S, T\} = \{F_1, F_2\}$ ,  $s = |\partial S|$ , and  $t = |\partial T|$ , and denote the vertices of  $G_S$  by  $u_i$ 's and those of  $G_T$  by  $v_j$ 's.

By Proposition 3.4, for  $t \geq 1$ , any negative edge in  $G_S$  has size at most  $t + 1$ . This last bound can be improved a bit in many cases, given that  $\Delta \geq 6$ , as shown below.

**Lemma 4.1.** *If  $\Delta \geq 6$  and  $t \geq 3$  then  $\Delta = 6$  and, in  $\overline{G}_S$ ,  $\deg \equiv 6$  and any edge has size  $t$ .*

*Proof.* By Lemma 3.1, any positive edge of  $\overline{G}_S$  has size at most  $t$ .

Suppose there is a negative edge  $\bar{e}$  in  $\overline{G}_S$  of size  $t + 1$ . By Lemma 3.2, any disk face of  $\overline{G}_S$  is even sided, and so  $\overline{G}_S$  has a vertex  $u_i$  of degree at most 4 by Lemma 2.2(b). If  $u_i$  has  $p$  positive and  $n$  negative local edges in  $\overline{G}_S$  then  $\deg_{\overline{G}_S}(u_i) = p + n \leq 4$  and so the degree of  $u_i$  in  $G_S$  satisfies the relations

$$6t \leq \Delta \cdot t = \deg_{G_S}(u_i) \leq p \cdot t + n \cdot (t + 1) = (p + n)t + n \leq 4t + 4,$$

whence  $t \leq 2$ , which is not the case.

Therefore any edge of  $\overline{G}_S$  has size at most  $t$ , so if  $u$  is any vertex of  $\overline{G}_S$  with  $p'$  positive and  $n'$  negative local edges, then again the degree of  $u$  in  $G_S$  satisfies the relations

$$6t \leq \Delta \cdot t = \deg_{G_S}(u) \leq p' \cdot t + n' \cdot t = (p' + n')t;$$

thus  $\deg_{\overline{G}_S}(u) = p' + n' \geq 6$ , and hence  $\deg \equiv 6$  in  $\overline{G}_S$  by Lemma 2.2(a). Since equality must then hold throughout the above relations, it follows that  $\Delta = 6$  and each edge of  $\overline{G}_S$  has size  $t$ .  $\square$

The *jumping number* of the graphs  $G_S$  and  $G_T$  was introduced in [2, §2]. For  $\Delta = 6$  the jumping number is one, which means that if the  $\Delta$  points of intersection between the circles  $\partial_i S (= u_i)$  and  $\partial_j T (= v_j)$  are labeled consecutively as  $x_1, x_2, \dots, x_\Delta$  around  $\partial_i S$ , then these points appear consecutively around  $\partial_j T$  in the same order  $x_1, x_2, \dots, x_\Delta$  when read in some direction. We will refer to this corresponding distribution of labels around the vertices of  $G_S$  and  $G_T$  as the *jumping number one condition*, or *JN1 condition* for short.

**4.1. The generic cases  $s, t \geq 3$ .** By Lemma 4.1,  $\Delta = 6$  and, in  $\overline{G}_S, \overline{G}_T$ ,  $\deg \equiv 6$  and all edges have size  $t, s$ , respectively; in particular, for any label  $1 \leq j \leq t$  ( $1 \leq i \leq s$ ), each vertex  $w$  of  $G_S$  ( $G_T$ , resp.) has 6 local edges with label  $j$  ( $i$ , resp.) at  $w$ , which give rise to the 6 local edges around  $w$  in  $\overline{G}_S$  ( $\overline{G}_T$ , resp.). The JN1 condition now implies that if  $e_1, e_2, e_3, e_4, e_5, e_6$  are the local edges with label  $j$  at  $u_i$ , as shown in Fig. 13, then these local edges appear with label  $i$  around  $v_j$  as shown in Fig. 13, up to reflection about a diameter of  $v_j$ ; and, by the parity rule, any local edge around  $v_j$  has the opposite sign of the corresponding local edge around  $u_i$ .

**Lemma 4.2.** *The cases  $\Delta = 6$  and  $s, t \geq 3$  do not occur.*

*Proof.* Assume  $s, t \geq 3$ , so that  $\Delta = 6$  and, in  $\overline{G}_S, \overline{G}_T$ ,  $\deg \equiv 6$ , each edge of  $\overline{G}_S, \overline{G}_T$  has size  $t, s$ , respectively, and hence all faces are triangles by

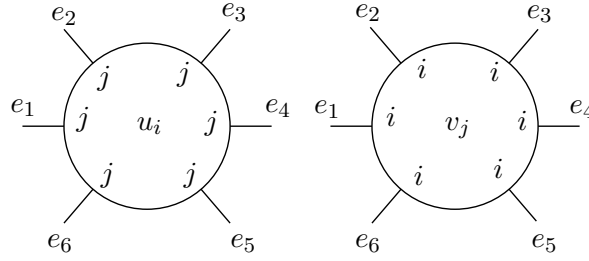


FIGURE 13.

Lemma 2.2(a). We will say that a vertex in  $\overline{G}_S, \overline{G}_T$  is of *type*  $(p, n)$  if it has  $p$  positive and  $n$  negative local edges, where  $p + n = 6$ .

Suppose some vertex  $u_i$  of  $\overline{G}_S$  is of type  $(p, n)$ . By the parity rule, each vertex  $v_j$  of  $\overline{G}_T$  has  $p$  negative and  $n$  positive local edges with label  $i$  at  $v_j$ ; thus, by our remarks above, any vertex of  $\overline{G}_T$  has  $p$  negative and  $n$  positive local edges and is therefore of type  $(n, p)$ . By a similar argument, any vertex of  $\overline{G}_S$  is of type  $(p, n)$ . Exchanging the roles of  $S$  and  $T$  if necessary, we may assume that  $(p, n)$  is one of the pairs  $(6, 0), (5, 1), (4, 2), (3, 3)$ .

If  $(p, n) = (6, 0)$  then every edge of  $\overline{G}_T$  is negative, which is impossible since all the faces of  $\overline{G}_T$  are triangles, and not all edges around a triangle face can be negative. Therefore  $(p, n) = (5, 1), (4, 2), (3, 3)$ , and so each graph  $\overline{G}_S, \overline{G}_T$  has at least one positive edge  $\overline{e}_S, \overline{e}_T$ , of size  $t, s$ , respectively.

Now, by Lemma 3.1, the edge orbits of  $\overline{e}_S, \overline{e}_T$  produce subgraphs isomorphic to the graph in Fig. 2(b) (thick edges only), and  $s, t$  are even. If  $\overline{G}_S$  has loop edges then  $\overline{G}_T$  must have a negative edge  $\overline{e}$  which induces the identity permutation; as  $|\overline{e}| = s$ , it follows that every vertex of  $\overline{G}_S$  has an incident loop edge, and hence that the subgraph of  $\overline{G}_S$  generated by the edge orbits of  $\overline{e}_T$  and  $\overline{e}$  is a union of components each isomorphic to the graph of Fig. 17(a). Therefore, since  $\deg \equiv 6$  in  $\overline{G}_S$ , if the graph  $\overline{G}_S$  has loop edges then it must be of the type shown in Fig. 14(a), where the thick edges represent the orbits of  $\overline{e}_T$ . A similar conclusion holds for  $\overline{G}_T$  whenever it has any loop edges.

If  $\overline{G}_S$  has no loop edges then we contradict Lemma 3.1, since each vertex of  $\overline{G}_S$  has  $p \geq 3$  positive local edges. Thus  $\overline{G}_S$  has loop edges and so it is a graph of the type shown in Fig. 14(a). Consider the vertices  $u, u'$  of  $\overline{G}_S$  indicated in Fig. 14(a), which lie in adjacent edge orbits of  $\overline{e}_T$ . If  $u$  and  $u'$  have opposite parity then  $u$  is of type  $(2, 4)$ , which is not the case. Therefore  $u$  and  $u'$  have the same parity and hence are of type  $(p, n) = (4, 2)$ , and the signs of the local edges as read consecutively around  $u$  are of the form  $- - + + + +$ . By the parity rule and the JN1 condition, the signs of the local edges as read consecutively around each vertex of  $\overline{G}_T$  are then of the form  $+ + - - - -$ .

Let  $v, v'$  be the vertices in some edge orbit  $c$  of  $\overline{e}_S$ ; then the two negative edges around, say, the vertex  $v$ , which are not on  $c$ , must both lie on the

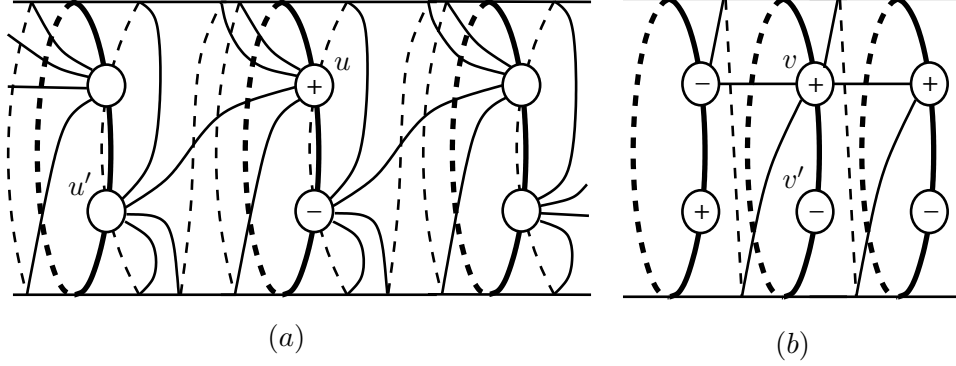


FIGURE 14.

same side of  $c$  (see Fig. 14(b)), which implies that not both  $v, v'$  can have incident loop edges and hence that  $\overline{G}_T$  has no loop edges by our preceding arguments. But then the two positive local edges at  $v$  must lie on the other side of  $c$ , as shown in Fig. 14(b), so that  $\deg_{\overline{G}_T}(v') \leq 4$ , contradicting the fact that  $\deg \equiv 6$  in  $\overline{G}_T$ . The lemma follows.  $\square$

**4.2. The cases  $s = 2, t \geq 3$ .** By Lemma 4.1,  $\Delta = 6$  and, in  $\overline{G}_S$ ,  $\deg \equiv 6$  and all the edges have size  $t$ ; also, recall that any negative edge of  $\overline{G}_T$  has size at most  $s + 1 = 3$ . In these cases  $\overline{G}_S$  is combinatorially isomorphic to the graph shown in Fig. 15 (cf [2, Lemma 5.2]), with vertices  $u_1, u_2$  and edges labeled  $\overline{e}_i, 1 \leq i \leq 6$ , and  $|\overline{e}_i| = t$ .

As  $\overline{e}_1, \overline{e}_2$  are positive loop edges in  $\overline{G}_S$ , it follows from Lemma 3.1 that  $t$  is even, so  $t \geq 4$ . We set  $\varepsilon = +1$  if  $u_1, u_2$  have the same parity (ie, if  $S$  is polarized), and  $\varepsilon = -1$  otherwise (if  $S$  is neutral). Then, for  $3 \leq i \leq 6$ , the edges  $\overline{e}_i$  have the same sign as  $\varepsilon$ .

Let  $\sigma_1, \sigma_2$  be the permutations induced by the edges  $\overline{e}_1, \overline{e}_2$ , respectively; notice that  $\overline{e}_3, \overline{e}_4, \overline{e}_5, \overline{e}_6$  all induce the same permutation  $\sigma$ . Using the generic labeling scheme of Fig. 15 (for some integers  $1 \leq \alpha, \beta \leq t$ ) we can see that  $\sigma_1(x) \equiv 1 - x$ ,  $\sigma_2(x) \equiv \alpha + \beta - x$ , and  $\sigma(x) \equiv \alpha + \varepsilon - \varepsilon \cdot x \pmod{t}$  for all  $1 \leq x \leq t$ . As  $\alpha + \varepsilon = \sigma(t) = \beta$ , we can write  $\sigma_2(x) \equiv 2\alpha + \varepsilon - x \pmod{t}$ .

The JN1 condition now implies that the local edges around  $u_i$  and  $v_j$  for  $i = 1, 2$  and  $j = 1, \dots, t$  are distributed as shown in Fig. 16 (up to reflections of the vertices). In the figure, the edges labeled  $e_\ell, \ell = 1, 2$ , or  $e_3, \dots, e_6$  are edges in the corresponding collections  $\overline{e}_k$  of  $\overline{G}_S$ , and represent the same edges in both graphs.

Notice also that some of the local edges around  $v_j$  in Fig. 16 may come from distinct parallel edges of  $G_T$ , since  $\deg \equiv 6$  need not hold in  $\overline{G}_T$ .

**Lemma 4.3.** *The cases  $s = 2$  and  $t \geq 3$  do not occur.*

*Proof.* Recall that  $t$  is even, so  $t \geq 4$ . Let  $\Gamma$  be the subgraph of  $\overline{G}_T$  generated by the edge orbits of  $\overline{e}_1$  and the  $\overline{e}_i$ 's,  $3 \leq i \leq 6$ .



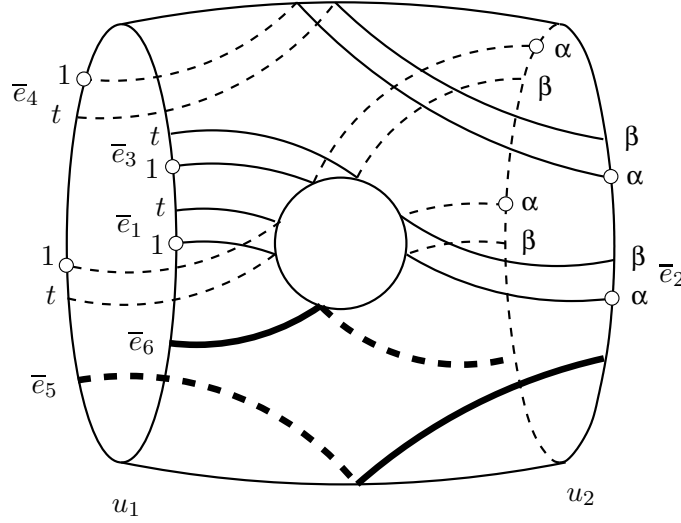


FIGURE 15.

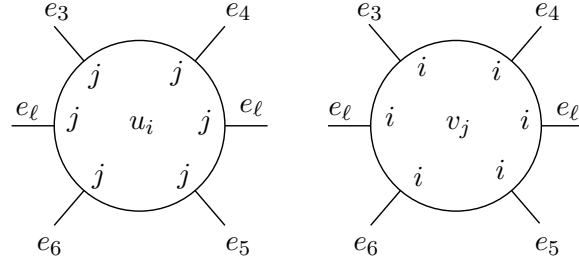


FIGURE 16.

Observe that  $\sigma = \sigma_1$  iff  $\sigma = \sigma_2$  iff  $\varepsilon = +1$  and  $\alpha \equiv 0 \pmod{t}$ . So, if  $\sigma = \sigma_1 = \sigma_2$  then  $\overline{G}_T$  is isomorphic to the subgraph of  $G_T$  generated by the edge orbits of  $\overline{e}_1$ , and hence each edge in  $\overline{G}_T$  is negative of size  $6 > s+2 = 4$ , contradicting the fact that any negative edge in  $\overline{G}_T$  can have size at most  $s+1 = 3$ . Therefore  $\sigma \neq \sigma_1, \sigma_2$ .

If  $\sigma = \text{id}$  then  $\Gamma$  is a union of components each isomorphic to the graph shown in Fig. 17(a), which violates the JN1 condition. Therefore  $\sigma \neq \text{id}$ .

Consider any two consecutive cycle edge orbits  $\gamma, \gamma'$  of  $\overline{e}_1$  in  $G_T$ , with opposite parity pairs of vertices  $v, v'$  and  $w, w'$ , respectively, and denote by  $A$  the annular region of  $T$  they cobound (see Fig. 17(b)). Let  $E$  be the collection of edges from  $\overline{e}_i$ ,  $3 \leq i \leq 6$ , that lie in  $A$ . Since  $\sigma \neq \sigma_1, \text{id}$ , none of the edges in  $E$  are loop edges nor parallel to the edges in  $\gamma, \gamma'$ , hence any such edge has one endpoint on a vertex of  $\gamma$  and the other on a vertex of  $\gamma'$ .

Now, by the JN1 condition, each vertex  $v, v', w, w'$  has local edges arising from the edges in  $E$ . Suppose  $a$  is edge of  $E$ , say with one endpoint on  $v$  and the other on  $w$ . If  $b$  is any edge of  $E$  with one endpoint on  $v'$  then,

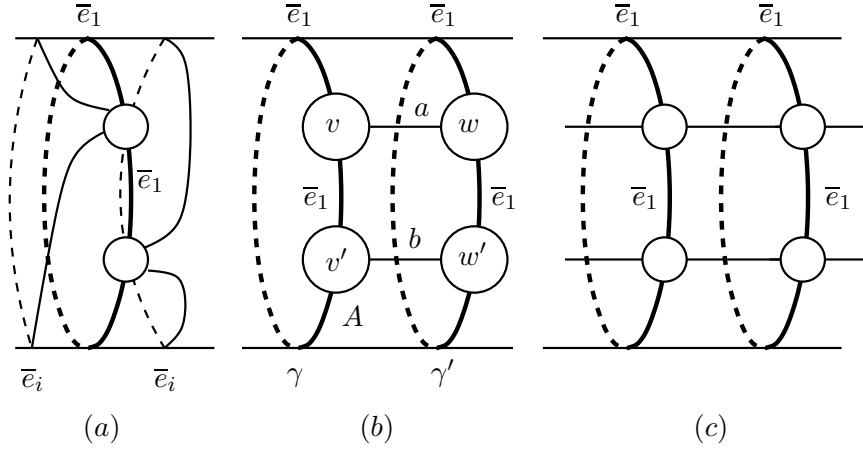


FIGURE 17.

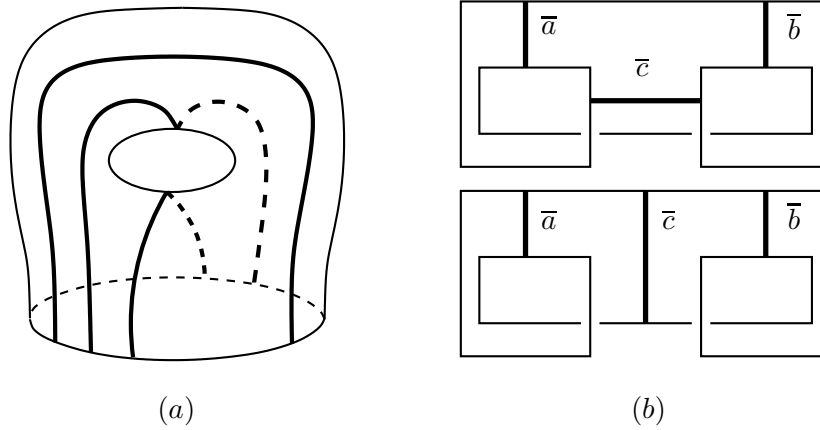


FIGURE 18.

by the parity rule, the other endpoint of  $b$  must lie on  $w'$  (see Fig. 17(b)). It follows that the subgraph  $\Gamma$  of  $\bar{G}_T$  is isomorphic to the graph shown Fig. 17(c), where necessarily each horizontal edge has size  $4 = s + 2$  and, by Lemma 2.1(b), consists of one edge from each collection  $\bar{e}_i$ ,  $3 \leq i \leq 6$ . Since any negative edge of  $\bar{G}_T$  can have size at most  $s + 1 = 3$ , the horizontal edges of  $\bar{G}_T$  must be positive, hence  $\varepsilon = -1$  by the parity rule and so both  $S$  and  $T$  are neutral.

Thus, any positive edge of  $\bar{G}_T$  has size 4 and hence its edges cobound three S-cycle faces in  $G_T$ , the outermost two of which locally lie on the same side of  $S$  and, by Lemma 2.1(b), have non parallel boundary circles in the surface  $S \cup I_{1,2}$  or  $S \cup I_{2,1}$ , as the case may be. Therefore, by Lemma 2.3,  $S$  is generated by an essential once punctured Klein bottle  $P$ .

Isotope  $P$  in  $M$  so as to intersect  $T$  transversely in essential graphs. We may assume that  $S$  is isotoped accordingly, so that the new graph  $S \cap T \subset T$  is essential and coincides with the frontier of a small regular neighborhood of the essential graph  $P \cap T \subset T$ ; in particular, the arguments above apply to the new graphs  $S \cap T \subset T$  and  $S \cap T \subset S$ . As  $P$  has at most two isotopy classes of negative edges and at most one isotopy class of positive edges (cf [5, Lemma 11] or [9, §2]), the reduced graph  $\overline{G}_P$  of  $G_P = P \cap T \subset P$  must be isomorphic to one of the graphs in Fig. 18(b), where the edges  $\overline{a}, \overline{b}$  are negative and  $\overline{c}$  is positive. Since all the edges of the reduced graph  $S \cap T \subset S$  have size  $t$  by Lemma 4.1, the edges of  $\overline{G}_P$  all have size  $t$  too (cf § 2.4). But then it is not hard to see that any negative edge in  $\overline{G}_P$  induces the identity permutation, which implies that any negative edge of the reduced graph of  $S \cap T \subset S$  also induces the identity permutation, contradicting our arguments above on the permutation  $\sigma$ . The lemma follows.  $\square$

#### 4.3. The cases $s = 1, t \geq 3$ .

**Lemma 4.4.** *The cases  $s = 1, t \geq 3$  do not occur.*

*Proof.* If  $s = 1$  then, as  $\deg \equiv 6$  in  $\overline{G}_S$  by Lemma 4.1,  $\overline{G}_S$  is isomorphic to the graph in Fig. 18(a), and hence all its edges induce the same permutation  $x \mapsto 1 - x \pmod t$ . Thus, if  $\overline{e}$  is any edge of  $\overline{G}_S$ , then  $\overline{G}_T$  is isomorphic to the subgraph of  $G_T$  generated by the cycle edge orbits of  $\overline{e}$ , and so in  $\overline{G}_T$  each edge is negative of size  $3 = s + 2$ , contradicting the fact that any negative edge in  $\overline{G}_T$  can have size at most  $s + 1 = 2$ .  $\square$

**4.4. Proof of Theorem 1.1.** Suppose  $(M, T_0)$  is not cabled and  $(F_1, \partial F_1), (F_2, \partial F_2) \subset (M, T_0)$  are  $\mathcal{K}$ -incompressible tori with boundary slopes at distance  $\Delta \geq 6$ . We set  $\{S, T\} = \{F_1, F_2\}$ , with  $s = |\partial S|$  and  $t = |\partial T|$ . Then  $1 \leq s, t \leq 2$  by Lemmas 4.2, 4.3, and 4.4, so it only remains to check that  $\Delta \leq 8$ . The case  $s = t = 1$  is impossible by the parity rule, and there are three more cases to consider.

**Case 1.**  $s = t = 2$  and  $S$  is polarized.

Then  $T$  is neutral and all the edges of  $G_T$  are negative. Hence  $\overline{G}_T$  has at most 4 edges, each of size at most  $s + 1 = 3$ , and  $\deg \leq 4$  in  $\overline{G}_T$ , so the degree of  $v_1$  in  $G_T$  satisfies the relations  $2\Delta = s \cdot \Delta = \deg_{G_T}(v_1) \leq 4 \cdot 3 = 12$ . Thus  $\Delta = 6$  (and  $\deg \equiv 4$  in  $\overline{G}_T$ , with each edge of  $\overline{G}_T$  of size  $s + 1 = 3$ ).

**Case 2.**  $s = t = 2$  and both  $S$  and  $T$  are neutral.

Then, in either graph  $\overline{G}_S, \overline{G}_T$ , any vertex has at most 4 negative edges and either 0 or 2 positive local edges (see Fig. 15), hence by Lemma 2.1(b) any local positive edge has size at most 4, while any negative edge has size at most 2. Therefore, if  $p, n$  are the number of positive and negative local edges of  $\overline{G}_S$  at  $u_1$ , respectively, then  $p \leq 2$  and  $n \leq 4$ , so the degree of  $u_1$  in  $G_S$  satisfies the relations  $2\Delta = s \cdot \Delta = \deg_{G_S}(u_1) \leq p \cdot 4 + n \cdot 2 \leq 16$ , and so  $\Delta \leq 8$ .

**Case 3.**  $s = 1$  and  $t = 2$ .

By the parity rule, since  $S$  is polarized then  $T$  is neutral and all edges of  $G_T$  negative, so  $\overline{G}_T$  has at most 4 edges, each of size at most  $s + 1 = 2$ . Hence  $\deg \leq 4$  in  $\overline{G}_T$ , and so the degree of  $v_1$  in  $G_T$  satisfies the relations  $\Delta = s \cdot \Delta = \deg_{G_T}(v_1) \leq 4 \cdot 2 \leq 8$ .  $\square$

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